Eur. Phys. J. C 10, 639–661 (1999)
Digital Object Identifier (DOI) 10.1007/s100529900054

THE EUROPEAN PHYSICAL JOURNAL C

SU(N) elastic rescattering in B and D decays

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Received: 17 December 1998 / Revised version: 17 February 1999 / Published online: 28 September 1999

Abstract. The treatment of elastic final-state interactions (FSIs) under a symmetry group is presented. The proposed model is based on Watson's theorem, i.e. on symmetry properties of the **S**-matrix and on its unitarity. This theorem provides an easy way to introduce rescattering effects by defining final-state interactions mixing matrices. A symmetry group fixes the structure of such mixing matrices, and the passage from one group to another is studied (for example, SU(2) to SU(3)). Mixings among two charmless pseudoscalar decay product states will be systematically analyzed. Finally, these mixing matrices will be used on quark diagram parametrizations of B and D decay amplitudes. This will have some important consequences on the definition of quark diagrams. It will be argued that these diagrams should not contain any FSI effects, i.e. they should be real (except for CKM factors). FSIs are then introduced at the hadronic level, by mixing basic quark diagram topologies.

1 Introduction

In this work, we will present a method for implementing final state interactions (FSIs or rescattering), i.e. the strong interactions between weak-decay products. These FSIs will be treated as elastic, and special care will be taken to define this concept. In particular, elasticity under a symmetry group will be defined as a special case of a generalized elasticity concept. This work has to be understood as a first step beyond the trivial treatment of FSIs, where FSIs are introduced as elastic under SU(2). The decays we have in mind are the B and D decays to two charmless pseudoscalars. In these B decays, CP violation is expected to occur. To be able to extract the values of the relevant parameters from experiments in order to compare them to standard-model values, we must dispose of an appropriate parametrization. Quark diagrams are usually thought to be appropriate for such a goal, but, as we will see, these quark diagrams should be properly defined in order to be of any use. In all this, FSIs play no fundamental role; they just mix up final states. Therefore, it is necessary to treat them to reach the underlying dynamics. The interesting point is that the model we propose to treat elastic FSIs, based on the **S**-matrix and Watson's theorem, will point towards a specific definition of the quark diagrams. A lot of papers exist on this subject; some of them are listed in the bibliography.

The decays we are considering proceed via the weak decay of a b or $\overline{c}$ quark. We will treat the weak interaction at the lowest order. The strong interactions are involved in the three following processes: they renormalize the weak interaction, they confine quarks into hadrons and they determine the asymptotic out states (FSI). Obviously, these two last manifestations of the strong interactions are a priori difficult to distinguish, because the out states can be considered as completely hadronized only when they no longer interact. The definition of FSI will be based on the following consideration: only hadrons, and not quarks enter the **S**-matrix. Consequently, FSIs will be defined as the (strong) interactions between hadrons. A typical decay process like $\overline{D^0} \to K^+\pi^-$ is a heavy quark $\overline{c}$ decaying "quickly" followed by the hadronization. This produces an intermediate real hadron state denoted inside accolades: for example $\overline{D^0} \to \{P_1P_2\}$. Then these hadrons interact by the FSI towards the final state: $\{P_1P_2\} \to K^+\pi^-$. This picture is quite schematic and we could say as well that we define intermediate decay amplitudes $\overline{D^0} \to \{P_1P_2\}$ (also qualified as bare) as free of any FSI effects. In other words, FSIs factorize from weak bare decay amplitudes. These bare amplitudes have no absorptive part since they must be real except for CKM factors (equivalently, their behavior under CP is simply $CP\Big(\overline{D^0} \to \{P_1P_2\}\Big) = \Big(\overline{D^0} \to \{P_1P_2\}\Big)^*$).

To summarize, the model will be based on three main points. (1) Unitarity of the **S**-matrix for a given set of rescattering channels, (2) the identification of bare amplitudes (elementary processes) as the part that get complex conjugated under CP. These two points will then imply that (3) FSI are treated as elastic among the chosen set of rescattering channels. Usually, when adding phases to isospin amplitudes to introduce FSI, one is implicitly considering that the **S**-matrix is unitary when restricted to a set of rescattering channels belonging to the same isomultiplet. We will extend this to more general sets of rescattering channels. This model can be characterized by

the way bare amplitudes are identified. This is a hypothesis, which is strictly equivalent to the general elasticity hypothesis as soon as the **S**-matrix is unitary. The validity of the present approach is discussed in the conclusion. Note that other hpropositions exist for the identification of bare amplitudes; see for instance the K-matrix formalism, which modelize an inelastic approach to the treatment of FSIs.

From this picture, we will naturally introduce quark diagrams (QDs) at the bare level. FSIs are then viewed as mixings of these bare amplitudes. Since bare amplitudes must be real, these QDs are defined as real. The important point is that by defining QDs at the bare level, they are characterized by basic topological configurations. This in turn is very important if we are to relate these QDs to elementary dynamical processes. Basic topologies are then mixed by FSI. We will develop all this further in the text.

For a given final state, we cannot have arbitrary intermediate states. FSIs being strong interactions, these intermediate states must have the same charge, strangeness, $\cdots$. Also, the available energy will determine the set of coupled open channels for a given set of quantum numbers. Among these coupled states, we will only consider two-pseudoscalar states. Thus we are neglecting transitions between these PP states and many-particle states, vector-meson states $\cdots$. This will be used when demonstrating Watson's theorem.

The approximate invariance under flavor exchange of the strong interaction implies very severe constraints on the decay amplitudes and on FSI. At the B or D mass, SU(2) or SU(3) are expected not to be badly broken. As we will see extensively, working under a symmetry group fixes the set of coupled states, this set being bigger under SU(3) than under SU(2). The symmetry group also fixes the structure of the couplings of these states. These couplings (or mixings) will be called SU(N) elastic ($N = 2, 3, \cdots$). One immediate question is to find a link between a SU(2) description and a SU(3) description of FSI, and this will be thoroughly carried out. Phenomenologically, it is sometimes questionable to treat SU(3) mixings as elastic; one example detailed at the end of this paper is the well-known SU(3) prediction $\overline{D^0} \to K^0\overline{K^0} = 0$, which can be lifted by an SU(3) breaking in the FSI. On the other hand the SU(2) restriction may be too strong, since we neglect many possible rescattering channels. An intermediate way is proposed in this work, by distinguishing elasticity from elasticity under a symmetry group.

Let us first recall how FSIs are usually treated when working under a flavor symmetry group.

1.1 SU(N) analyses of B and D decays

The B and D decays we wish to describe are those into two charmless pseudoscalars. We will work under SU(2) or SU(3), at the lowest order in the electroweak interaction.

1.1.1 Isospin analysis

Let us analyze the decays $\overline{D^0}$ to $K^+\pi^-$ and $K^0\pi^0$ under SU(2). The well-known isospin analysis leads to the following parametrization of the physical decay amplitudes:

$$\begin{cases} \left(\overline{D^0} \to K^+\pi^-\right) = A^{3/2} + A^{1/2}, \\ \left(\overline{D^0} \to K^0\pi^0\right) = \frac{1}{\sqrt{2}}\left(2A^{3/2} - A^{1/2}\right), \end{cases} \tag{1}$$

where these isospin amplitudes correspond to $A^T = \langle T\,|H_W = 1|\,1/2\rangle$. CKM elements are not explicitly written. These amplitudes contain the weak interaction at the lowest order, and the whole strong interaction, including FSI. The usual procedure to take FSI into account is to associate phases with the isospin amplitudes by

$$\begin{cases} A^{3/2} = \mathrm{e}^{\mathrm{i}\delta_{3/2}} A_{\mathrm{b}}^{3/2}, \\ A^{1/2} = \mathrm{e}^{\mathrm{i}\delta_{1/2}} A_{\mathrm{b}}^{1/2}. \end{cases} \tag{2}$$

We can therefore identify the bare amplitudes for these decays:

$$\begin{cases} \left(\overline{D^0} \to \{K^+\pi^-\}\right) = A_{\mathrm{b}}^{3/2} + A_{\mathrm{b}}^{1/2}, \\ \left(\overline{D^0} \to \left\{K^0\pi^0\right\}\right) = \frac{1}{\sqrt{2}}\left(2A_{\mathrm{b}}^{3/2} - A_{\mathrm{b}}^{1/2}\right). \end{cases} \tag{3}$$

From these bare amplitudes, we can reintroduce FSI using a matrix procedure:

$$\begin{pmatrix} \left(\overline{D^0} \to K^+\pi^-\right) \\ \left(\overline{D^0} \to K^0\pi^0\right) \end{pmatrix} = M^{\mathrm{SU}(2)} \begin{pmatrix} \left(\overline{D^0} \to \{K^+\pi^-\}\right) \\ \left(\overline{D^0} \to \left\{K^0\pi^0\right\}\right) \end{pmatrix}, \tag{4}$$

with $M^{\mathrm{SU}(2)}$ given by

$$M^{\mathrm{SU}(2)} = \frac{1}{3}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\delta_{3/2}} + 2\mathrm{e}^{\mathrm{i}\delta_{1/2}} & \sqrt{2}\left(\mathrm{e}^{\mathrm{i}\delta_{3/2}} - \mathrm{e}^{\mathrm{i}\delta_{1/2}}\right) \\ \sqrt{2}\left(\mathrm{e}^{\mathrm{i}\delta_{3/2}} - \mathrm{e}^{\mathrm{i}\delta_{1/2}}\right) & 2\mathrm{e}^{\mathrm{i}\delta_{3/2}} + \mathrm{e}^{\mathrm{i}\delta_{1/2}} \end{pmatrix}. \tag{5}$$

This matrix method is strictly equivalent to the usual procedure (2). However, it is now apparent that FSIs are introduced as mixings between the $\{K^+\pi^-\}$ and $\left\{K^0\pi^0\right\}$ intermediate states.

1.1.2 SU(3) analysis

The SU(3) analyses of B and D decays into two uncharmed pseudoscalars are given in the Appendix. Let us consider the following set of decays:

$$\begin{cases} \left(\overline{D}^0 \to K^+\pi^-\right) = \left(-4A^{27} + 4A^8 - 2B^8\right), \\ \left(\overline{D}^0 \to K^0\pi^0\right) = \frac{1}{\sqrt{2}}\left(-6A^{27} - 4A^8 + 2B^8\right), \\ \left(\overline{D}^0 \to K^0\eta_8\right) = \frac{1}{\sqrt{6}}\left(-6A^{27} - 4A^8 + 2B^8\right); \end{cases} \tag{6}$$

again CKM elements are not written explicitly. The usual procedure to take FSI into account in this SU(3) context is simply

$$\begin{cases} A^{27} \to \mathrm{e}^{\mathrm{i}\delta_{27}} A_{\mathrm{b}}^{27} = A^{27}, \\ X^8 \to \mathrm{e}^{\mathrm{i}\delta_8} X_{\mathrm{b}}^8 = X^8 \quad \text{with} \quad X = A, B. \end{cases} \tag{7}$$

We can therefore identify bare decays as

$$M^{\mathrm{SU}(3)} = \frac{1}{5}\begin{pmatrix} 2\mathrm{e}^{\mathrm{i}\delta_{27}} + 3\mathrm{e}^{\mathrm{i}\delta_8} & \frac{3}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\delta_{27}} - \mathrm{e}^{\mathrm{i}\delta_8}\right) & \sqrt{\frac{3}{2}}\left(\mathrm{e}^{\mathrm{i}\delta_{27}} - \mathrm{e}^{\mathrm{i}\delta_8}\right) \\ \frac{3}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\delta_{27}} - \mathrm{e}^{\mathrm{i}\delta_8}\right) & \frac{1}{2}\left(7\mathrm{e}^{\mathrm{i}\delta_{27}} + 3\mathrm{e}^{\mathrm{i}\delta_8}\right) & -\frac{\sqrt{3}}{2}\left(\mathrm{e}^{\mathrm{i}\delta_{27}} - \mathrm{e}^{\mathrm{i}\delta_8}\right) \\ \sqrt{\frac{3}{2}}\left(\mathrm{e}^{\mathrm{i}\delta_{27}} - \mathrm{e}^{\mathrm{i}\delta_8}\right) & -\frac{\sqrt{3}}{2}\left(\mathrm{e}^{\mathrm{i}\delta_{27}} - \mathrm{e}^{\mathrm{i}\delta_8}\right) & \frac{1}{2}\left(9\mathrm{e}^{\mathrm{i}\delta_{27}} + \mathrm{e}^{\mathrm{i}\delta_8}\right) \end{pmatrix}. \tag{10}$$

$$\begin{cases} \left(B^0 \to \left\{K^+\pi^-\right\}\right) = V_{ub}^* V_{us}\left(-4A_{\mathrm{b}}^{27} - A_{\mathrm{b}}^8 - B_{\mathrm{b}}^8 - C_{\mathrm{b}}^8\right) + V_{cb}^* V_{cs}\left(-C_b^{8c}\right) + V_{tb}^* V_{ts}\left(-C_{\mathrm{b}}^{8t}\right), \\ \left(B^0 \to \left\{K^0\pi^0\right\}\right) = \frac{1}{\sqrt{2}}\left(V_{ub}^* V_{us}\left(-6A_{\mathrm{b}}^{27} + A_{\mathrm{b}}^8 + B_{\mathrm{b}}^8 + C_{\mathrm{b}}^8\right) + V_{cb}^* V_{cs}\left(C_b^{8c}\right) + V_{tb}^* V_{ts}\left(C_{\mathrm{b}}^{8t}\right)\right), \\ \left(B^0 \to \left\{K^0\eta_8\right\}\right) = \frac{1}{\sqrt{6}}\left(V_{ub}^* V_{us}\left(-6A_{\mathrm{b}}^{27} + A_{\mathrm{b}}^8 + B_b^8 + C_{\mathrm{b}}^8\right) + V_{cb}^* V_{cs}\left(C_b^{8c}\right) + V_{tb}^* V_{ts}\left(C_{\mathrm{b}}^{8t}\right)\right). \end{cases} \tag{11}$$

$$\begin{cases} \left(\overline{D^0} \to \{K^+\pi^-\}\right) = \left(-4A_{\mathrm{b}}^{27} + 4A_{\mathrm{b}}^8 - 2B_{\mathrm{b}}^8\right), \\ \left(\overline{D^0} \to \left\{K^0\pi^0\right\}\right) = \frac{1}{\sqrt{2}}\left(-6A_{\mathrm{b}}^{27} - 4A_{\mathrm{b}}^8 + 2B_{\mathrm{b}}^8\right), \\ \left(\overline{D^0} \to \left\{K^0\eta_8\right\}\right) = \frac{1}{\sqrt{6}}\left(-6A_{\mathrm{b}}^{27} - 4A_{\mathrm{b}}^8 + 2B_{\mathrm{b}}^8\right). \end{cases} \tag{8}$$

And starting with these decompositions, we can reintroduce FSI using a matrix procedure:

$$\begin{pmatrix} \left(\overline{D^0} \to K^+\pi^-\right) \\ \left(\overline{D^0} \to K^0\pi^0\right) \\ \left(\overline{D^0} \to K^0\eta_8\right) \end{pmatrix} = M^{\mathrm{SU}(3)} \begin{pmatrix} \left(\overline{D^0} \to \{K^+\pi^-\}\right) \\ \left(\overline{D^0} \to \left\{K^0\pi^0\right\}\right) \\ \left(\overline{D^0} \to \left\{K^0\eta_8\right\}\right) \end{pmatrix}, \tag{9}$$

with (see (10) on top of the page).

As for SU(2), we see that FSI effects reduce to some mixings among intermediate states. But a major difference arises: under SU(3), the $\left\{K^0\eta_8\right\}$ also mixes with $\{K\pi\}$ states. This mixing goes beyond SU(2) since $\{K\pi\}$ states and $\left\{K^0\eta_8\right\}$ are in different SU(2) representations.

The same matrix $M^{\mathrm{SU}(3)}$ can also be used to introduce FSI in other decay decompositions into matrix elements. For example, B^0 bare decays (see (11) on top of the page). And we can see that applying $M^{\mathrm{SU}(3)}$ is equivalent to the usual prescription (7) and $C_{\mathrm{b}}^8 \to \mathrm{e}^{\mathrm{i}\delta_8} C_{\mathrm{b}}^8 = C^8$.

1.2 Questions

Having written those matrix representations for FSI, the following questions can be addressed:

(1) *What is the underlying theoretical framework?* We would like to know precisely the hypotheses concerning this procedure. Also, the elasticity concept has to be properly defined. Finally, the properties of these M matrices like unitarity and symmetry should be explained. This section is based on Watson's theorem.

(2) *What are the SU(N) flavor symmetry implications?* We would like to find a systematic way to calculate mixing matrices like $M^{\mathrm{SU}(2)}$ and $M^{\mathrm{SU}(3)}$. The link between these two matrices will also be analyzed. The fact that the same matrix can be used for different sets of reactions will be explained.

(3) *What can we say about the use of quark diagrams?* We will argue that quark diagrams should be used to parametrize bare decays. In other words, we will parametrize physical decay amplitudes using FSI mixing matrices for the rescattering effects, and quark diagrams for the weak decays, their gluonic corrections (but no absorptive part) and the hadronizations.

These three points will be considered in the three following sections. In the last section, we will apply the advocated procedure to systematically analyze the B and D decays into two uncharmed pseudoscalars.

Let us summarize the general procedure we suggest in this paper. Watson's theorem implies that the physical decay amplitudes for a set of processes can be factorized into an FSI part and a bare part. We can then extract from the full weak amplitudes the FSI contributions by putting some intermediate states on-shell, and these states are hadron states entering the **S**-matrix. Bare amplitudes are then parametrized using quark diagrams, and FSIs are introduced using mixing matrices. Finally, dynamical (e.g. the choice of the set of coupled states) or symmetry (e.g. SU(2)) considerations will determine the form of these matrices.

2 Theoretical framework

2.1 Generalized Watson's theorem

Part of the following discussion is borrowed from [30] and [25]. Watson's theorem will allow us to single out the final-state interaction effects inside the physical weak-decay amplitudes of B or D. Remember that we are working to lowest order in the electroweak interaction. Let us begin by expressing the generalized Watson theorem. By W we denote the column vector formed with the weak amplitudes into the possible final states:

$$W = \left(B \to \pi\pi \; B \to \pi\pi\pi\pi \; B \to K\overline{K} \cdots\right)^{\mathrm{t}}. \tag{12}$$

S will be the **S**-matrix containing the coupling among these final states:

$$S = \begin{pmatrix} \pi\pi \to \pi\pi & \pi\pi \to \pi\pi\pi\pi & \pi\pi \to K\overline{K} & \cdots \\ \pi\pi\pi\pi \to \pi\pi & \pi\pi\pi\pi \to \pi\pi\pi\pi & \pi\pi\pi\pi \to K\overline{K} & \cdots \\ K\overline{K} \to \pi\pi & K\overline{K} \to \pi\pi\pi\pi & K\overline{K} \to K\overline{K} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{13}$$

The processes entering this S-matrix proceed dominantly via the strong interaction, the weak contribution being much smaller. Thus this matrix is block-diagonal, each block representing mixing among states of definite flavor quantum numbers. The important point is that W and

S are built from hadron states like $K, \pi, \eta, D, B, \cdots$; i.e. states which decay via electroweak interactions only.

The *generalized Watson theorem* then reads

$$W = \sqrt{S} W_{\rm b}, \qquad CP(W) = \sqrt{S} W_{\rm b}^*. \tag{14}$$

This means that FSI effects contained in $S^{1/2}$ factorize, leaving bare amplitudes $W_{\rm b}$, which contain no FSIs. These amplitudes will be written as

$$W_{\rm b} = \left(B \to \{\pi\pi\}\ B \to \{\pi\pi\pi\pi\}\ B \to \left\{K\overline{K}\right\} \cdots \right)^{\rm t}, \tag{15}$$

where the $\{\}$ denotes intermediate states. These bare amplitudes contain the weak decay of the heavy quark, with its gluonic corrections, and the hadronization of the intermediate hadron state, but no FSI effect. These rescattering effects are introduced as interactions between these intermediate hadron states using $S^{1/2}$. We can also say that this theorem allows one to extract from physical amplitudes the FSI part from the bare part, and this is done at the hadronic level. In other words, it is hadron states entering the **S**-matrix that are put on mass shell as intermediate states. This can also be interpreted as a renormalization of bare amplitudes induced by rescattering effects.

The complete demonstration is in the Appendix. The main features are as follows.

(1) Watson's theorem follows from the unitarity condition for the complete **S**-matrix built from W and S:

$$\mathbf{S}^\dagger \mathbf{S} = \mathbf{S}\mathbf{S}^\dagger = 1 \text{ with } \mathbf{S} = \begin{pmatrix} 1 & {\rm i}W^{\rm t} \\ {\rm i}CP(W) & S \end{pmatrix}. \tag{16}$$

Thus we can say that $W = S^{1/2} W_{\rm b}$ is a unitarization of weak bare decay amplitudes, since with the adjunction of the strong phases $S^{1/2}$, the full **S**-matrix is unitary.

(2) Bare amplitudes are identified as the part of physical amplitudes that get complex conjugated under CP. That is the main point, since this identification implies elasticity as we will see in the next paragraph. Inverting the argument, we want to build a model of FSI based on the elastic hypothesis, we are thus led to this identification.

2.2 Elasticity

Elasticity is equivalent to the unitarity of the strong **S**-matrix containing the coupling. The elasticity hypothesis is then hidden in the feature (1) above concerning unitarity of **S**. Indeed, as soon as **S** is unitary, S and $S^{1/2}$ are also unitary (see (98)). This implies that we have conservation of probability among the coupled channels:

$$W^\dagger W = W_{\rm b}^\dagger \sqrt{S}^\dagger \sqrt{S}\ W_{\rm b} = W_{\rm b}^\dagger W_{\rm b}. \tag{17}$$

If we note the intermediate states, i.e. states produced and not yet rescattered by $\{{\rm x}_i\}$ and final out states by ${\rm x}_i$, we can rewrite the strong **S**-matrix (13) as

$$S = \begin{pmatrix} \{x_1\} \to x_1 & \{x_2\} \to x_1 & \cdots & \{x_n\} \to x_1 \\ \{x_1\} \to x_2 & \{x_2\} \to x_2 & \cdots & \{x_n\} \to x_2 \\ \vdots & \vdots & \ddots & \vdots \\ \{x_1\} \to x_n & \{x_2\} \to x_n & \cdots & \{x_n\} \to x_n \end{pmatrix}. \tag{18}$$

Probability conservation can be expressed as

$$\sum_{i=1}^{n} \left\| (B \to x_i) \right\|^2 = \sum_{i=1}^{n} \left\| (B \to \{x_i\}) \right\|^2. \tag{19}$$

This is another expression of elasticity. It is clear that if we consider all the possible final states, **S** will be unitary. In practice, however, we consider couplings only among a subset of final states (for example, only $\pi\pi$ states), and thus we neglect many other possible mixings. We then impose the unitarity of a truncated **S**-matrix, limited to this subset of states. This is the elastic hypothesis; it is characterized by probability conservation among this subset of states.

The most important restriction we will impose on the mixings is to consider coupling between states of two pseudoscalars (in the s wave) only. Mixing with states containing vector mesons, or many particle states are thus neglected. This restriction is convenient in order to ensure a symmetric form for S. Indeed, S will be symmetric if the transition amplitudes are invariant under time reversal, and since a general state may catch a different phase than PP states under CP, they will not be considered.

2.2.1 FSI eigenchannels

Let us define some technical tools used in the rest of the paper:

(1) The basis of *eigenchannels* $|C_i\rangle$ where S is diagonal, with matrix elements

$$\left\| \langle C_i | S | C_j \rangle \right\|^2 = \delta_{ij} \quad \Rightarrow \quad \langle C_i | S | C_j \rangle = \delta_{ij} {\rm e}^{2{\rm i}\delta_{C_i}}. \tag{20}$$

Therefore these states C_i do not mix under rescattering. Elasticity becomes manifest in the unit norm, which in turn is equivalent to unitarity for S.

(2) These diagonal elements of S are the *strong phases*:

$$S_{\rm diag} \equiv \begin{pmatrix} {\rm e}^{2{\rm i}\delta_{C1}} & 0 & \cdots 0 \\ 0 & {\rm e}^{2{\rm i}\delta_{C2}} & \cdots 0 \\ \vdots & \vdots & \ddots \vdots \\ 0 & 0 & \cdots {\rm e}^{2{\rm i}\delta_{Cn}} \end{pmatrix}. \tag{21}$$

(3) Since the **S**-matrix is symmetric and unitary, we can diagonalize it using *a real orthogonal transformation* O:

$$S = O^{\rm t} S_{\rm diag} O. \tag{22}$$

The orthogonal diagonalizing matrix O also relates the eigenchannel basis to the physical one:

$$\left| \overrightarrow{C} \right\rangle \equiv \begin{pmatrix} |C_1\rangle \\ |C_2\rangle \\ \vdots \\ |C_n\rangle \end{pmatrix} = O \begin{pmatrix} \{x_1\} \\ \{x_n\} \\ \vdots \\ \{x_n\} \end{pmatrix} \equiv O \left| \overrightarrow{\{x\}} \right\rangle. \tag{23}$$

(4) The *mixing matrix* M is simply the square root of S, which appears in Watson's theorem, and is defined as

$$\begin{cases} M \equiv \sqrt{S} = O^{\mathrm{t}} \sqrt{S_{\mathrm{diag}}} O, \\ M_{\mathrm{diag}} \equiv \sqrt{S_{\mathrm{diag}}}. \end{cases} \tag{24}$$

(5) With these tools, we can give another derivation of Watson's theorem. Let us define some renormalized out eigenstates by

$$\left|\overrightarrow{C}\right\rangle \to \left|\overrightarrow{C}_{\mathrm{out}}\right\rangle = \sqrt{S_{\mathrm{diag}}} \left|\overrightarrow{C}\right\rangle, \tag{25}$$

and this is equivalent to (20)

$$\langle C_{i,\mathrm{out}} \mid C_{j,\mathrm{out}} \rangle = \langle C_i \left| S_{\mathrm{diag}} \right| C_j \rangle = \delta_{ij} \mathrm{e}^{2\mathrm{i}\delta_i}. \tag{26}$$

In the physical basis we have the following situation:

$$\begin{cases} \left|\overrightarrow{C}\right\rangle = O \left|\overrightarrow{\{x\}}\right\rangle, \\ \left|\overrightarrow{C}_{\mathrm{out}}\right\rangle = O \left|\overrightarrow{x}\right\rangle, \end{cases} \tag{27}$$

i.e. the development of the intermediate states in terms of intermediate eigenchannels is the same as the development of final asymptotic states in terms of out eigenchannels. Putting all this in equations gives

$$\left|\overrightarrow{x}\right\rangle = O^{\mathrm{t}} \left|\overrightarrow{C}_{\mathrm{out}}\right\rangle = O^{\mathrm{t}} \sqrt{S_{\mathrm{diag}}} \left|\overrightarrow{C}\right\rangle = O^{\mathrm{t}} \sqrt{S_{\mathrm{diag}}} O \left|\overrightarrow{\{x\}}\right\rangle, \tag{28}$$

and the final result is

$$\left|\overrightarrow{x}\right\rangle = \sqrt{S} \left|\overrightarrow{\{x\}}\right\rangle \equiv M \left|\overrightarrow{\{x\}}\right\rangle. \tag{29}$$

For the decay amplitudes, we recover Watson's theorem:

$$\begin{aligned} \overrightarrow{(B \to x)} &= \begin{pmatrix} (B \to x_1) \\ (B \to x_2) \\ \vdots \\ (B \to x_n) \end{pmatrix} = M \begin{pmatrix} (B \to \{x_1\}) \\ (B \to \{x_2\}) \\ \vdots \\ (B \to \{x_n\}) \end{pmatrix} \\ &= M \overrightarrow{(B \to \{x\})}. \end{aligned} \tag{30}$$

2.2.2 Application to D decays

The treatment of weak-decay amplitudes of D is the same as in the B case. We will write

$$\overrightarrow{(D \to x)} = M \overrightarrow{(D \to \{x\})}. \tag{31}$$

However, since FSIs in D decays proceed at a lower energy than in B decays, and since the number of open channels in D decays is much smaller than in B decays, the M-matrix structure is not necessarily the same for B and D decays, and the rescattering phases are different.

2.2.3 One rescattering channel

In the case of only one rescattering channel, writing $W_{\mathrm{b}} = W\mathrm{e}^{\mathrm{i}\gamma}$ and $S^{1/2} = \mathrm{e}^{\mathrm{i}\delta}$, $W = S^{1/2} W_{\mathrm{b}}$ is equivalent to the well-known result:

$$\begin{cases} W = W\mathrm{e}^{\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}\delta}, \\ CP\,(W) = W\mathrm{e}^{-\mathrm{i}\gamma}\mathrm{e}^{\mathrm{i}\delta}, \end{cases} \tag{32}$$

i.e. that under CP the weak phase γ is reversed and not the strong phase δ. The matrix version for many rescattering channels really appears as a simple generalization.

3 SU(N) flavor symmetry implications

3.1 SU(N) elasticity: Definition

A mixing will be elastic under SU(N) if

(1) the FSI eigenchannels C_i are definite states of SU(N).
(2) The orthogonal transformation is the Clebsch–Gordan coefficient matrix relating the physical states to the SU(N) states.
(3) The phases (M_{diag} matrix elements) depend on the representation of the corresponding eigenchannels only.

The point (2) has a direct and important consequence: working under a symmetry group *fixes the form of mixing matrices* (they are completely determined by the representation contents of the final states), and *imposes some restrictions on the mixings*. Indeed, a group of states coupled together under SU(N) is a group of states containing some common representations of SU(N). The decaying meson $(B, D, \cdots)$, on the other hand, fixes the energy scale at which FSIs take place, and thus determines the values of the strong phases.

In general, a symmetry group fixes the set of coupled channels, and this set increases with N. On the other hand, the symmetry breaking also increases with N. For example, in B decays, we can treat the mixing $K\pi$, $K\eta$ under SU(4), in order to include charmed meson channels, but the breaking of SU(4) at the B-mass energy is such that SU(4) elasticity is expected to be inappropriate.

Terminology To properly define elasticity, we can distinguish the following concepts:

(i) *Pure elastic transitions* like $\{K^+\pi^-\} \to K^+\pi^-$ with just a phase as amplitude.
(ii) *SU(N) elasticity*, for which we have elasticity in a basis of eigenchannels corresponding to SU(N) states.
(iii) *Elasticity*, for which we can define some general eigenchannels C_i by a set of mixing parameters (see 112). This is sometimes considered as inelastic since we can have, for example, eigenchannels with no specific isospin.

3.2 SU(2) analysis of $K^+\pi^-$, $K^0\pi^0$, $K^0\eta_8$

To accomplish the connection with Sect. 1, consider the system $K^+\pi^-$, $K^0\pi^0$, $K^0\eta_8$ and suppose we are working under SU(2). The SU(2) FSI eigenchannels are then

$$\begin{aligned} \begin{pmatrix} |C_1\rangle \\ |C_2\rangle \\ |C_3\rangle \end{pmatrix} &\equiv \begin{pmatrix} |3/2, -1/2\rangle \\ \left|1/2^{(1)}, -1/2\right\rangle \\ \left|1/2^{(2)}, -1/2\right\rangle \end{pmatrix} \\ &= \begin{pmatrix} -\sqrt{1/3} & -\sqrt{2/3} & 0 \\ -\sqrt{2/3} & \sqrt{1/3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \{K^+\pi^-\} \\ \{K^0\pi^0\} \\ \{K^0\eta_8\} \end{pmatrix}. \end{aligned} \tag{33}$$

This means that $K^+\pi^-$, $K^0\pi^0$ are mixed under SU(2), since they contain the same representations of isospin 3/2 and $1/2^{(1)}$, but $K^0\eta_8$, being in a different 1/2 representation, stays alone. The $O_{\mathrm{SU}(2)}$ matrix (Clebsch–Gordan coefficients) is block-diagonal, and so is $M^{\mathrm{SU}(2)}$. The phases entering $M^{\mathrm{SU}(2)}_{\mathrm{diag}}$, depending only on the eigenchannels SU(2) representations, are:

$$M^{\mathrm{SU}(2)}_{\mathrm{diag}} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\delta_{3/2}} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\delta^{(1)}_{1/2}} & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\delta^{(2)}_{1/2}} \end{pmatrix}. \tag{34}$$

The SU(2) elasticity is expressed in these eigenchannels as

$$\begin{cases} \left\|\langle 3/2,-1/2\,|S|\,3/2,-1/2\rangle\right\|^2 = 1 \\ \left\|\left\langle 1/2^{(1)},-1/2\,|S|\,1/2^{(1)},-1/2\right\rangle\right\|^2 = 1 \\ \left\|\left\langle 1/2^{(2)},-1/2\,|S|\,1/2^{(2)},-1/2\right\rangle\right\|^2 = 1 \end{cases} \Rightarrow \begin{cases} \langle 3/2,-1/2\,|S|\,3/2,-1/2\rangle = \mathrm{e}^{2\mathrm{i}\delta_{3/2}}, \\ \left\langle 1/2^{(1)},-1/2\,|S|\,1/2^{(1)},-1/2\right\rangle = \mathrm{e}^{2\mathrm{i}\delta^{(1)}_{1/2}}, \\ \left\langle 1/2^{(2)},-1/2\,|S|\,1/2^{(2)},-1/2\right\rangle = \mathrm{e}^{2\mathrm{i}\delta^{(2)}_{1/2}}. \end{cases} \tag{35}$$

The M-matrix is then calculated as $M^{\mathrm{SU}(2)} = O^{\mathrm{t}}_{\mathrm{SU}(2)} M^{\mathrm{SU}(2)}_{\mathrm{diag}} O_{\mathrm{SU}(2)}$, and we finally have

$$M^{\mathrm{SU}(2)} = \begin{pmatrix} \frac{1}{3}\left(\mathrm{e}^{\mathrm{i}\delta_{3/2}} + 2\mathrm{e}^{\mathrm{i}\delta^{(1)}_{1/2}}\right) & \frac{\sqrt{2}}{3}\left(\mathrm{e}^{\mathrm{i}\delta_{3/2}} - \mathrm{e}^{\mathrm{i}\delta^{(1)}_{1/2}}\right) & 0 \\ \frac{\sqrt{2}}{3}\left(\mathrm{e}^{\mathrm{i}\delta_{3/2}} - \mathrm{e}^{\mathrm{i}\delta^{(1)}_{1/2}}\right) & \frac{1}{3}\left(2\mathrm{e}^{\mathrm{i}\delta_{3/2}} + \mathrm{e}^{\mathrm{i}\delta^{(1)}_{1/2}}\right) & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\delta^{(2)}_{1/2}} \end{pmatrix}, \tag{36}$$

which is the same matrix (for the $K\pi$ sector) as in Sect. 1 (see (5)).

3.2.1 Intermediate and asymptotic states

At this point, we can repeat the discussion of the preceding section and define out isospin eigenchannels:

$$\begin{pmatrix} |C_1\rangle = |3/2,-1/2\rangle \\ |C_2\rangle = |1/2,-1/2\rangle \end{pmatrix} \to \begin{pmatrix} |C_{1,\mathrm{out}}\rangle = |3/2,-1/2\rangle_{\mathrm{out}} \\ |C_{2,\mathrm{out}}\rangle = |1/2,-1/2\rangle_{\mathrm{out}} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\delta_{3/2}}\,|3/2,-1/2\rangle \\ \mathrm{e}^{\mathrm{i}\delta_{1/2}}\,|1/2,-1/2\rangle \end{pmatrix}, \tag{37}$$

and for the physical states we write

$$\begin{pmatrix} \{K^+\pi^-\} \\ \{K^0\pi^0\} \end{pmatrix} = O^{\mathrm{t}}_{\mathrm{SU}(2)} \begin{pmatrix} |C_1\rangle = |3/2,-1/2\rangle \\ |C_2\rangle = |1/2,-1/2\rangle \end{pmatrix}, \tag{38}$$

$$\begin{pmatrix} K^+\pi^- \\ K^0\pi^0 \end{pmatrix} = O^{\mathrm{t}}_{\mathrm{SU}(2)} \begin{pmatrix} |C_{1,out}\rangle = |3/2,-1/2\rangle_{\mathrm{out}} \\ |C_{2,out}\rangle = |1/2,-1/2\rangle_{\mathrm{out}} \end{pmatrix},$$

so the link between intermediate and asymptotic states is given by

$$\begin{pmatrix} K^+\pi^- \\ K^0\pi^0 \end{pmatrix} = M^{\mathrm{SU}(2)} \begin{pmatrix} \{K^+\pi^-\} \\ \{K^0\pi^0\} \end{pmatrix}. \tag{39}$$

This shows once again that SU(2) fixes the structure of $M^{\mathrm{SU}(2)}$ and that only the representation contents of the final states is relevant. This in turn implies that the same matrix is appropriate for B and D decays. Of course, the phases $\delta_{3/2}$ and $\delta_{1/2}$ can be different since their specific values is a dynamical question (they depend on the energy available, i.e. the mass of the decaying meson). So, for example, we can write

$$\begin{pmatrix} (B^0 \to K^+\pi^-) \\ (B^0 \to K^0\pi^0) \end{pmatrix} = M^{\mathrm{SU}(2)} \begin{pmatrix} (B^0 \to \{K^+\pi^-\}) \\ (B^0 \to \{K^0\pi^0\}) \end{pmatrix}. \tag{40}$$

3.2.2 Decay amplitudes

As we have shown in Sect. 1, by applying this $M^{\mathrm{SU}(2)}$ matrix on the isospin decompositions into bare amplitudes (3), we find again (1). This is a general principle. The usual procedure to take into account SU(N) FSIs in a SU(N) bare amplitude decomposition is to add phases to the SU(N) bare amplitudes according to their SU(N) representations. As we have said, applying the SU(N) mixing matrix on SU(N) bare decompositions is equivalent, i.e. the usual prescription is equivalent to a mixing of states, and the M-matrices provide a clear representation of these mixings.

Let us illustrate this fact in the example of $K^+\pi^-$, $K^0\pi^0$ under SU(2). From

$$\left|H_W \quad \overline{D^0}\right\rangle = \sqrt{\frac{1}{3}}\,|3/2\rangle - \sqrt{\frac{2}{3}}\,|1/2\rangle \tag{41}$$

and (33), we found the isospin decomposition (3) with $A^T_{\mathrm{b}} \sim \langle T \mid T\rangle$. The point is to note that the same orthogonal transformation is used in the calculation of decomposition and of mixing matrices (this remains valid under any SU(N)). To find the decompositions in terms of full amplitudes, just replace the intermediate states $|T\rangle$ by out states $\mathrm{e}^{\mathrm{i}\delta_T}\,|T\rangle$, and this is strictly equivalent to renormalize $A^T_{\mathrm{b}} \to A^T \sim \mathrm{e}^{\mathrm{i}\delta_T}\,\langle T \mid T\rangle$.

3.3 SU(3) analysis of $K^+\pi^-$, $K^0\pi^0$, $K^0\eta_8$

Under SU(3), the same group of states mixes completely because they all contain the same 27 and 8:

$$\begin{pmatrix} |C_1\rangle \\ |C_2\rangle \\ |C_3\rangle \end{pmatrix} \equiv \begin{pmatrix} |27,3/2,-1/2,1\rangle \\ |27,1/2,-1/2,1\rangle \\ |8_S,1/2,-1/2,1\rangle \end{pmatrix} = \begin{pmatrix} -\sqrt{1/3} & -\sqrt{2/3} & 0 \\ -\sqrt{1/15} & \sqrt{1/30} & -\sqrt{9/10} \\ -\sqrt{3/5} & \sqrt{3/10} & \sqrt{1/10} \end{pmatrix}$$

$$\times \begin{pmatrix} \{K^+\pi^-\} \\ \{K^0\pi^0\} \\ \{K^0\eta_8\} \end{pmatrix}, \tag{42}$$

where SU(3) states are specified as $|rep., T, T_3, Y\rangle$. $\mathrm{M}_{\mathrm{diag}}^{\mathrm{SU(3)}}$ is given by

$$M_{\mathrm{diag}}^{\mathrm{SU(3)}} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\delta_{27}} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\delta_{27}} & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\delta_8} \end{pmatrix}. \tag{43}$$

From this orthogonal transformation and $M_{\mathrm{diag}}^{\mathrm{SU(3)}}$, we immediately recover (10):

$$M^{\mathrm{SU(3)}} = \frac{1}{5} \begin{pmatrix} 2\mathrm{e}^{\mathrm{i}\delta_{27}} + 3\mathrm{e}^{\mathrm{i}\delta_8} & \frac{3}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\delta_{27}} - \mathrm{e}^{\mathrm{i}\delta_8}\right) & \sqrt{\frac{3}{2}}\left(\mathrm{e}^{\mathrm{i}\delta_{27}} - \mathrm{e}^{\mathrm{i}\delta_8}\right) \\ \frac{3}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\delta_{27}} - \mathrm{e}^{\mathrm{i}\delta_8}\right) & \frac{1}{2}\left(7\mathrm{e}^{\mathrm{i}\delta_{27}} + 3\mathrm{e}^{\mathrm{i}\delta_8}\right) & -\frac{\sqrt{3}}{2}\left(\mathrm{e}^{\mathrm{i}\delta_{27}} - \mathrm{e}^{\mathrm{i}\delta_8}\right) \\ \sqrt{\frac{3}{2}}\left(\mathrm{e}^{\mathrm{i}\delta_{27}} - \mathrm{e}^{\mathrm{i}\delta_8}\right) & -\frac{\sqrt{3}}{2}\left(\mathrm{e}^{\mathrm{i}\delta_{27}} - \mathrm{e}^{\mathrm{i}\delta_8}\right) & \frac{1}{2}\left(9\mathrm{e}^{\mathrm{i}\delta_{27}} + \mathrm{e}^{\mathrm{i}\delta_8}\right) \end{pmatrix}.$$

The same matrix $M^{\mathrm{SU(3)}}$ can also be used for other decays, since it is determined by the representation contents of the final states only. For example:

$$\begin{cases} (B_s \to \{K^-\pi^+\}) \\ \left(B_s \to \left\{\overline{K}^0\pi^0\right\}\right) \\ \left(B_s \to \left\{\overline{K}^0\eta_8\right\}\right) \end{cases} \quad \text{or} \quad \begin{cases} \left(\overline{D}^0 \to \{K^+\pi^-\}\right) \\ \left(\overline{D}^0 \to \left\{K^0\pi^0\right\}\right) \\ \left(\overline{D}^0 \to \left\{K^0\eta_8\right\}\right) \end{cases}. \tag{44}$$

We have thus recovered and explained the results of Sect. 1.2.

3.4 Probability conservation

We can use probability conservation to characterize the difference between SU(2) and SU(3) elasticity (see (111)). Probability conservation can be expressed under SU(2) by (see (45) on top of the next page) and under SU(3), in a less restrictive way, by (see (46) on top of the next page).

3.5 Links between SU(N) and SU($N\pm1$)

3.5.1 Principle

As we have seen above, an elastic mixing under SU(N) is not in general elastic under SU($N-1$). Suppose we have the matrices M^{SU}(N) and $M^{\mathrm{SU}(N-1)}$. This latter matrix is block diagonal, since it mixes only some subsets of states. Therefore, to go from SU($N-1$) towards SU(N) elastic mixings, we will have to add to $M^{\mathrm{SU}(N-1)}$ some extra mixing between different sets of SU($N-1$) coupled states. These new mixings are not completely arbitrary: they must be compatible with the SU($N-1$) included in SU(N). Such extra mixings will be parametrized by mixing parameters $\alpha, \beta, \cdots$, and we will obtain a generalized mixing matrix $M^{\mathrm{gen.}}(\alpha, \beta, \cdots)$. Finally, for a specific value of $\alpha, \beta, \cdots$, this matrix will correspond to $M^{\mathrm{SU}(N)}$.

We will not describe the most general case, but we will take again the system $K^+\pi^-$, $K^0\pi^0$, $K^0\eta_8$ and perform the transition from a SU(2) description to a SU(3) description. The discussion in the general case is then straightforward.

3.5.2 From SU(2) to SU(3)

Step 1 *Building of the most general mixing among $K^+\pi^-$, $K^0\pi^0$, $K^0\eta_8$ compatible with isospin.*

As we have seen, (33) defines isospin eigenchannels for SU(2) FSI. The only possible extra mixing is between the two 1/2 eigenchannels, since we want to keep isospin as a good quantum number for FSI eigenchannels. This extra mixing can be parametrized by a general 2×2 orthogonal matrix (112):

$$\begin{pmatrix} |C_1\rangle \\ |C_2\rangle \\ |C_3\rangle \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} |3/2, -1/2\rangle \\ \left|1/2^{(1)}, -1/2\right\rangle \\ \left|1/2^{(2)}, -1/2\right\rangle \end{pmatrix}. \tag{47}$$

In terms of physical intermediate states, this gives

$$\begin{pmatrix} |C_1\rangle \\ |C_2\rangle \\ |C_3\rangle \end{pmatrix} \equiv \begin{pmatrix} -\sqrt{1/3} & -\sqrt{2/3} & 0 \\ -\sqrt{2/3}\cos\alpha & \sqrt{1/3}\cos\alpha & -\sin\alpha \\ -\sqrt{2/3}\sin\alpha & \sqrt{1/3}\sin\alpha & \cos\alpha \end{pmatrix} \times \begin{pmatrix} \{K^+\pi^-\} \\ \{K^0\pi^0\} \\ \{K^0\eta_8\} \end{pmatrix}. \tag{48}$$

This equation defines new eigenchannels and the corresponding orthogonal transformation $O(\alpha)$. Note that these new eigenchannels keep isospin as a good quantum number (C_1: 3/2, C_2 and C_3: 1/2). $M_{\mathrm{diag}}^{\mathrm{general}}$ in this basis is given by

$$M_{\mathrm{diag}}^{\mathrm{general}} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\delta_{C1}} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\delta_{C2}} & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\delta_{C3}} \end{pmatrix}. \tag{49}$$

We can calculate $M^{\mathrm{general}}(\alpha) = O(\alpha)^{\mathrm{t}} M_{\mathrm{diag}}^{\mathrm{general}} O(\alpha)$ (see (50) on top of the next page) where $s = \sin\alpha$ and $c = \cos\alpha$.

For each value of the mixing parameter α, $M^{\mathrm{general}}(\alpha)$ parametrizes an elastic mixing among $K^+\pi^-$, $K^0\pi^0$, $K^0\eta_8$ compatible with isospin (but inelastic for SU(2)).

Step 2 *SU(3) mixing appears as a special case of the general mixing $M^{\mathrm{general}}(\alpha)$.*

Indeed, SU(3) mixings are obtained from the SU(2) ones by introducing an extra mixing compatible with SU(2), since the isospin group is a subgroup of SU(3) (a SU(3) elastic eigenchannel could not be a mixture of 3/2 and 1/2 isospin states). The SU(3) limit is now easy to get: for $\cos\alpha = (1/10)^{1/2}$ and $\sin\alpha = (9/10)^{1/2}$, O($\alpha$)

$$\mathrm{SU}(2)\begin{cases}\left\|\left(\overline{D}\to\left\{K^+\pi^-\right\}\right)\right\|^2+\left\|\left(\overline{D}\to\left\{K^0\pi^0\right\}\right)\right\|^2=\left\|\left(\overline{D}\to K^+\pi^-\right)\right\|^2+\left\|\left(\overline{D}\to K^0\pi^0\right)\right\|^2,\\ \left\|\left(\overline{D}\to\left\{K^0\eta_8\right\}\right)\right\|^2=\left\|\left(\overline{D}\to K^0\eta_8\right)\right\|^2\end{cases}\tag{45}$$

$$\mathrm{SU}(3)\begin{cases}\left\|\left(\overline{D}\to\left\{K^+\pi^-\right\}\right)\right\|^2+\left\|\left(\overline{D}\to\left\{K^0\pi^0\right\}\right)\right\|^2+\left\|\left(\overline{D}\to\left\{K^0\eta_8\right\}\right)\right\|^2\\ \quad=\left\|\left(\overline{D}\to K^+\pi^-\right)\right\|^2+\left\|\left(\overline{D}\to K^0\pi^0\right)\right\|^2+\left\|\left(\overline{D}\to K^0\eta_8\right)\right\|^2.\end{cases}\tag{46}$$

$$\frac{1}{3}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\delta_{C1}}+2c^2\mathrm{e}^{\mathrm{i}\delta_{C2}}+2s^2\mathrm{e}^{\mathrm{i}\delta_{C3}} & \sqrt{2}\left(\mathrm{e}^{\mathrm{i}\delta_{C1}}-c^2\mathrm{e}^{\mathrm{i}\delta_{C2}}-s^2\mathrm{e}^{\mathrm{i}\delta_{C3}}\right) & \sqrt{6}sc\left(\mathrm{e}^{\mathrm{i}\delta_{C2}}-\mathrm{e}^{\mathrm{i}\delta_{C3}}\right)\\ \sqrt{2}\left(\mathrm{e}^{\mathrm{i}\delta_{C1}}-c^2\mathrm{e}^{\mathrm{i}\delta_{C2}}-s^2\mathrm{e}^{\mathrm{i}\delta_{C3}}\right) & 2\mathrm{e}^{\mathrm{i}\delta_{C1}}+c^2\mathrm{e}^{\mathrm{i}\delta_{C2}}+s^2\mathrm{e}^{\mathrm{i}\delta_{C3}} & -\sqrt{3}sc\left(\mathrm{e}^{\mathrm{i}\delta_{C2}}-\mathrm{e}^{\mathrm{i}\delta_{C3}}\right)\\ \sqrt{6}sc\left(\mathrm{e}^{\mathrm{i}\delta_{C2}}-\mathrm{e}^{\mathrm{i}\delta_{C3}}\right) & -\sqrt{3}sc\left(\mathrm{e}^{\mathrm{i}\delta_{C2}}-\mathrm{e}^{\mathrm{i}\delta_{C3}}\right) & 3(s^2\mathrm{e}^{\mathrm{i}\delta_{C2}}+c^2\mathrm{e}^{\mathrm{i}\delta_{C3}})\end{pmatrix},\tag{50}$$

$$M^{\mathrm{SU(2\ in\ 3)}}=\begin{pmatrix}\frac{\mathrm{e}^{\mathrm{i}\delta_{27}^{3/2}}}{3}+\frac{\mathrm{e}^{\mathrm{i}\delta_{27}^{1/2}}}{15}+\frac{3\mathrm{e}^{\mathrm{i}\delta_8^{1/2}}}{5} & \frac{1}{\sqrt{2}}\left(\frac{2\mathrm{e}^{\mathrm{i}\delta_{27}^{3/2}}}{3}-\frac{\mathrm{e}^{\mathrm{i}\delta_{27}^{1/2}}}{15}-\frac{3\mathrm{e}^{\mathrm{i}\delta_8^{1/2}}}{5}\right) & \frac{\sqrt{6}}{10}\left(\mathrm{e}^{\mathrm{i}\delta_{27}^{1/2}}-\mathrm{e}^{\mathrm{i}\delta_8^{1/2}}\right)\\ \frac{1}{\sqrt{2}}\left(\frac{2\mathrm{e}^{\mathrm{i}\delta_{27}^{3/2}}}{3}-\frac{\mathrm{e}^{\mathrm{i}\delta_{27}^{1/2}}}{15}-\frac{3\mathrm{e}^{\mathrm{i}\delta_8^{1/2}}}{5}\right) & \frac{2\mathrm{e}^{\mathrm{i}\delta_{27}^{3/2}}}{3}+\frac{\mathrm{e}^{\mathrm{i}\delta_{27}^{1/2}}}{30}+\frac{3\mathrm{e}^{\mathrm{i}\delta_8^{1/2}}}{10} & \frac{\sqrt{3}}{10}\left(\mathrm{e}^{\mathrm{i}\delta_8^{1/2}}-\mathrm{e}^{\mathrm{i}\delta_{27}^{1/2}}\right)\\ \frac{\sqrt{6}}{10}\left(\mathrm{e}^{\mathrm{i}\delta_{27}^{1/2}}-\mathrm{e}^{\mathrm{i}\delta_8^{1/2}}\right) & \frac{\sqrt{3}}{10}\left(\mathrm{e}^{\mathrm{i}\delta_8^{1/2}}-\mathrm{e}^{\mathrm{i}\delta_{27}^{1/2}}\right) & \frac{1}{10}\left(9\mathrm{e}^{\mathrm{i}\delta_{27}^{1/2}}+\mathrm{e}^{\mathrm{i}\delta_8^{1/2}}\right)\end{pmatrix}.\tag{52}$$

equals the orthogonal matrix of SU(3) Clebsch–Gordan (42). This means that the three channels C_i tend towards the SU(3) states $|27,3/2,-1/2,1\rangle$, $|27,1/2,-1/2,1\rangle$ and $|8_S,1/2,-1/2,1\rangle$. Then the corresponding phases tend towards the SU(3) phases: $\mathrm{e}^{\mathrm{i}\delta_{C1}},\mathrm{e}^{\mathrm{i}\delta_{C2}}\to\mathrm{e}^{\mathrm{i}\delta_{27}}$ and $\mathrm{e}^{\mathrm{i}\delta_{C3}}\to\mathrm{e}^{\mathrm{i}\delta_8}$. We have thus completed the passage from SU(2) to SU(3).

Remarks: (1) Note that it is sometimes necessary to introduce a $P=diag(\pm1,\cdots,\pm1)$ matrix in order to have $P\cdot O(\alpha)\to O^{\mathrm{SU(3)}}$, since the latter depends on phase conventions. On the other hand, the M-matrix, being physical, is always phase-convention independent. In other words, orthogonal transformations differing by a P-matrix give the same M-matrix.

(2) In SU(2), the mixing of $K\pi$ states with state $K^0\eta_8$ is neglected. We have shown in this section that the extra mixing needed to treat the mixings among $K^+\pi^-$, $K^0\pi^0$, $K^0\eta_8$ under SU(3) elasticity is quite big (the two channels 1/2 get nearly inverted). At the cost of one unknown mixing parameter α, we can use (50), which is compatible with both SU(2) and SU(3) to introduce a "small" mixing between $K\pi$ and $K^0\eta_8$. This will be done in Sect. 5.

3.5.3 SU(2) in SU(3)

Still in the same example, we will illustrate another link between SU(3) and SU(2). A very interesting form for the mixing matrix is built from the following diagonal form:

$$M_{\mathrm{diag}}^{\mathrm{SU(2\ in\ 3)}}=\begin{pmatrix}\mathrm{e}^{\mathrm{i}\delta_{27}^{3/2}} & 0 & 0\\ 0 & \mathrm{e}^{\mathrm{i}\delta_{27}^{1/2}} & 0\\ 0 & 0 & \mathrm{e}^{\mathrm{i}\delta_8^{1/2}}\end{pmatrix},\tag{51}$$

i.e. we distinguish isospin in SU(3); the three phases are different. From this matrix, by applying the orthogonal SU(3) transformation, we obtain (see (52) on top of the page).

This form can be very useful in phenomenological analyses, since it is an easy way to implement SU(3) breaking in the FSI . We can recover the SU(2) and SU(3) limits straightforwardly. If we identify $\mathrm{e}^{\mathrm{i}\delta_{27}^{3/2}}$ and $\mathrm{e}^{\mathrm{i}\delta_{27}^{1/2}}$, we find again $M^{\mathrm{SU(3)}}$. On the other hand, if we identify $\mathrm{e}^{\mathrm{i}\delta_{27}^{3/2}}$ with $\mathrm{e}^{\mathrm{i}\delta_{3/2}}$, $\mathrm{e}^{\mathrm{i}\delta_{27}^{1/2}}$ and $\mathrm{e}^{\mathrm{i}\delta_8^{1/2}}$ with $\mathrm{e}^{\mathrm{i}\delta_{1/2}}$, we find the following mixing:

$$M_{\mathrm{SU(2)}}^{\mathrm{modified}}=\begin{pmatrix}\frac{1}{3}\left(\mathrm{e}^{\mathrm{i}\delta_{3/2}}+2\mathrm{e}^{\mathrm{i}\delta_{1/2}}\right) & \frac{\sqrt{2}}{3}\left(\mathrm{e}^{\mathrm{i}\delta_{3/2}}-\mathrm{e}^{\mathrm{i}\delta_{1/2}}\right) & 0\\ \frac{\sqrt{2}}{3}\left(\mathrm{e}^{\mathrm{i}\delta_{3/2}}-\mathrm{e}^{\mathrm{i}\delta_{1/2}}\right) & \frac{1}{3}\left(2\mathrm{e}^{\mathrm{i}\delta_{3/2}}+\mathrm{e}^{\mathrm{i}\delta_{1/2}}\right) & 0\\ 0 & 0 & \mathrm{e}^{\mathrm{i}\delta_{1/2}}\end{pmatrix},\tag{53}$$

which is built from the SU(2) orthogonal transformation (33) with a modified $M_{\mathrm{diag}}^{\mathrm{SU(2)}}$ (see (34)). This modified form is obtained from the identification: $\delta_{1/2}^{(1)}=\delta_{1/2}^{(2)}=\delta_{1/2}$. We can easily explain this: the SU(3) orthogonal transformation has the structure $O(\alpha)=O^{\mathrm{extra}}O^{\mathrm{SU(2)}}$ (see (48)). Then $M_{\mathrm{SU(2)}}^{\mathrm{modified}}=O^{\mathrm{SU(2)},t}O^{\mathrm{extra},t}M_{\mathrm{diag}}^{\mathrm{SU(2)}}O^{\mathrm{extra}}O^{\mathrm{SU(2)}}$and with $\delta_{1/2}^{(1)}=\delta_{1/2}^{(2)}$, O^{extra} simplifies, leaving the SU(2) transformation.

3.5.4 Concluding remarks

This section is of theoretical and practical importance. On the theoretical side, we have defined SU(N) elasticity as a special case of the general elasticity concept.

On the practical side, we have shown how to build mixing matrices explicitly. We have obtained four different mixing matrices for the system $\left\{K^+\pi^-,K^0\pi^0,K^0\eta_8\right\}$: $M^{\mathrm{general}}(\alpha),M^{\mathrm{SU(3)}},M^{\mathrm{SU(2)}}$ and $M^{\mathrm{SU(2\ in\ 3)}}$ with its

limit $M^{\text{modified}}_{\text{SU(2)}}$. We can now choose to use any form, independently of the parametrization chosen for the bare decay amplitudes. This illustrates the power of this matrix method for treating FSI. Since we have factorized FSI from bare amplitudes inside physical amplitudes, these two aspects can be analyzed independently.

We stress again that all this can be repeated for other mixings, other flavor symmetry groups and other meson decays.

4 Quark diagrams

The B and D decays can be parametrized with quark diagram (QD) amplitudes. The ultimate goal of QDs is to test the standard model. To achieve this, we must compare a calculated value of an amplitude to its measured value. We will show that *by defining quark diagrams as free of any FSI effect, they are well-defined in terms of basic topologies*, and thus allow in principle to reach such a goal.

Using Watson' theorem, physical amplitudes can be decomposed into bare amplitudes and FSI matrices. We are thus naturally led to the following parametrization:

$$\text{Physical decay amplitudes} \overset{\text{Watson's theorem}}{\rightarrow} \begin{cases} \text{Bare amplitudes: Quark diagrams} \\ \text{FSI effects: Mixing matrices} \end{cases}$$

We will now analyze the consequences.

4.1 Quark diagrams as bare amplitudes

Since quark diagrams are free of any FSI effects, they are real, except for CKM phases. In other words, QDs are defined at the level of bare amplitudes. They contain the weak decay of the heavy quark, the hadronization and some gluonic renormalizations of the weak current (without absorptive part).

FSIs are introduced as interactions between hadrons, using FSI phases. In this way, we avoid the difficulties (if not the inconsistencies), of on-shell quarks since we are always working with on-shell hadrons, i.e. states entering the **S**-matrix. The treatment of FSI at the hadronic level is ultimately justified by the **S**-matrix hadronic structure.

4.2 Quark-diagram topologies and FSI topology mixings

Quark diagrams are built from quark lines and W lines. The resulting topologies (or "shapes") of these diagrams are the usual tree T, color-suppressed C, annihilation A, exchange E, penguin P and penguin annihilation PA diagrams (see [2–11]). These can also be defined from two basic topologies (a bubble, representing the quark–gluon sea, with a W inside and two bubbles connected by a W), from which we extract real hadrons (Fig. 1).

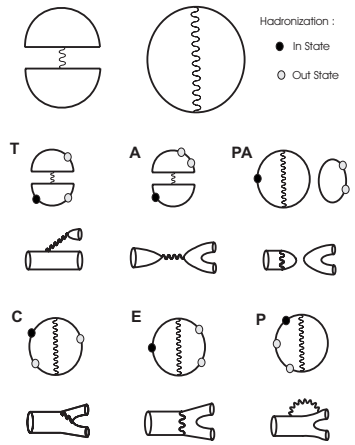

Fig. 1. The two basic topologies and quark diagrams extractions

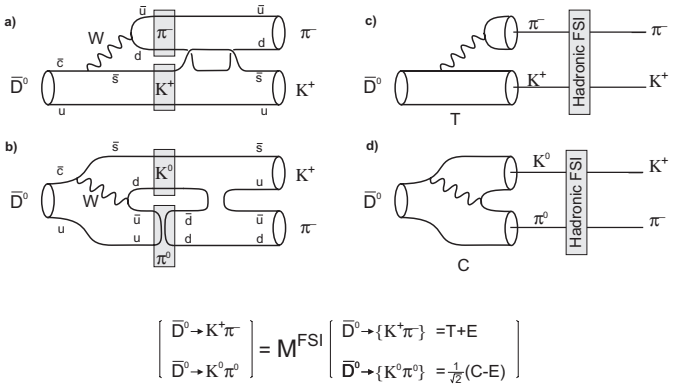

Fig. 2. Illustration of the Watson theorem using quark diagrams: FSIs are viewed as mixings of basic quark diagram topologies

The assertion that QDs allow dynamical considerations when defined at the bare level comes from the well-known fact that final-state interactions mix the different topologies. In particular, any scheme like factorization should be carried at the level of these "bare" QDs. Furthermore, helicity suppression of A is valid only if A is a bare amplitude, since otherwise it could contain other basic topologies than A.

The establishment of the QD parametrizations of bare decay amplitudes is explained in Appendix 2, where the link between SU(3) bare amplitudes and QDs is also written. The QD parametrizations of B and D decays are given in Appendix 3.

Discussion In Fig. 2 we have drawn two possible diagrams ((a) and (b)) contributing to $\left(\overline{D}^0 \to K^+\pi^-\right)$. The scheme of introducing QDs at the bare level and FSI as hadronic mixing matrices is depicted in the passage from (a) and (b) to (c) and (d), i.e. by the identification of the relevant hadronic intermediate states. These hadrons then interact, and this interaction is the FSI. Figures (c) and (d)

also show that basic quark diagrams automatically occur as bare topologies, and the resulting physical amplitude $\left(\overline{D}^0 \to K^+\pi^-\right)$ receives contributions from its QD topologies $\overline{D}^0 \to \{K^+\pi^-\} = \mathrm{T} + \mathrm{E}$, and also from some extra topologies (here, some C) coming from $\overline{D}^0 \to \left\{K^0\pi^0\right\}$.

If we had introduced quark diagrams at the physical level, we would have obtained for the physical amplitudes $\left(\overline{D}^0 \to K^+\pi^-\right) = T^{\text{full}} + E^{\text{full}}$. Now looking at the quark lines in Fig. 2b, we see that this diagram is topologically equivalent to the E topology, i.e. that it contributes to E^{full}. So this E^{full} amplitude, containing FSI effects, contains some C topology. Doing the same analysis with Fig. 2a, we can see that this diagram is a non-factorizable contribution to T^{full}. When introduced with FSI effects, T loses its factorization properties. By analyzing some other decays, one can easily see that an amplitude A^{full} receives contributions from other topologies than A. Thus A^{full} is not helicity suppressed.

In conclusion, this discussion shows that in order to have well-defined quark diagrams in terms of basic topologies, they should be defined at the bare level, without rescattering effects. We have also shown how some considerations like factorizability or helicity suppression collapse when QDs contain FSI effects.

Remark: The elastic hypothesis is to introduce these quark diagrams at the level of bare amplitudes. Other propositions exist, for example to introduce QDs at the level of different amplitudes in the K-matrix formalism (for this K-matrix formalism, see for example [13–18]).

4.3 Use of mixing matrices on QD decompositions

By using Watson's theorem we have shown that FSI and bare processes separate. We can thus analyze each part independently; for example,

$$\begin{cases} \textbf{Bare amplitudes}: \begin{cases} \mathrm{SU}(2), \mathrm{SU}(3), \cdots \text{ amplitudes} \\ \text{Quark diagrams} \to \\ \text{under SU}(2), \mathrm{SU}(3), \cdots \end{cases} \\ \textbf{FSI effects:} \\ \textbf{(Mixing matrices)} \begin{cases} \text{under SU}(2) \text{ with } M^{\mathrm{SU}(2)} \\ \text{under SU}(3) \text{ with } M^{\mathrm{SU}(3)} \\ \text{as a general mixing} \\ \text{with } M^{\text{general}}(\alpha, \beta, \cdots) \\ \text{under SU}(3) \text{ with SU}(2) \\ \text{specified with } M^{\mathrm{SU}(2 \text{ in } 3)} \\ \cdots \end{cases} \end{cases}$$

The physical decay amplitudes are obtained by applying the chosen mixing matrix on the chosen bare decay parametrizations. The next section will extensively illustrate this procedure.

5 Applications to B and D decays

The two pseudoscalar final states can be grouped into sets of coupled states under SU(3) (denoted inside {}) by considering conserved quantum numbers: isospin T and hypercharge Y. In fact, these sets correspond to the sets of definite T_3 and Y, since they completely mix. The only exception is $\left\{\pi^-\pi^0\right\}$ states (pure isospin 2) which do not mix with $\left\{K^-K^0, \pi^-\eta_8\right\}$(pure isospin 1). Repeating the same analysis, we can also find sets of coupled states under SU(2). The results for SU(3) and SU(2) mixing are

(a) $K\pi$ **and** $\overline{K}\pi$ **sets** $(Y = \pm 1, T_3 = \pm 1/2)$

$$\begin{aligned} \mathrm{A}: \mathrm{SU}(3) &: \left\{K^+\pi^-, K^0\pi^0, K^0\eta_8\right\} \\ \mathrm{SU}(2) &: \left\{K^+\pi^-, K^0\pi^0\right\}\left\{K^0\eta_8\right\}, \\ \mathrm{B}: \mathrm{SU}(3) &: \left\{K^0\pi^+, K^+\pi^0, K^+\eta_8\right\} \\ \mathrm{SU}(2) &: \left\{K^0\pi^+, K^+\pi^0\right\}\left\{K^+\eta_8\right\}, \\ \mathrm{C}: \mathrm{SU}(3) &: \left\{\overline{K^0}\pi^-, K^-\pi^0, K^-\eta_8\right\} \\ \mathrm{SU}(2) &: \left\{\overline{K^0}\pi^-, K^-\pi^0\right\}\left\{K^-\eta_8\right\}, \\ \mathrm{D}: \mathrm{SU}(3) &: \left\{K^-\pi^+, \overline{K^0}\pi^0, \overline{K^0}\eta_8\right\} \\ \mathrm{SU}(2) &: \left\{K^-\pi^+, \overline{K^0}\pi^0\right\}\left\{\overline{K^0}\eta_8\right\}. \end{aligned}$$

(b) $K\overline{K}, \pi\eta$ **sets** $(Y = 0, T_3 = \pm 1)$

$$\begin{aligned} E: \mathrm{SU}(3) &: \left\{K^-K^0, \pi^-\eta_8\right\}\left\{\pi^-\pi^0\right\} \\ \mathrm{SU}(2) &: \left\{K^-K^0\right\}\left\{\pi^-\eta_8\right\}\left\{\pi^-\pi^0\right\}, \\ \mathrm{F}: \mathrm{SU}(3) &: \left\{K^+\overline{K^0}, \pi^+\eta_8\right\}\left\{\pi^+\pi^0\right\} \\ \mathrm{SU}(2) &: \left\{K^+\overline{K^0}\right\}\left\{\pi^+\eta_8\right\}\left\{\pi^+\pi^0\right\}. \end{aligned}$$

(c) $K\overline{K}, \pi\eta, \pi\pi, \eta\eta$ **set** $(Y = 0, T_3 = 0)$

$$\mathrm{G}: \begin{array}{c} \left\{K^-K^+, K^0\overline{K^0}, \eta_8\eta_8, \pi^+\pi^-, \pi^0\pi^0, \pi^0\eta_8\right\} \\ \downarrow^{\mathrm{SU}(3)}_{\mathrm{SU}(2)} \\ \left\{K^-K^+, K^0\overline{K^0}\right\}\left\{\eta_8\eta_8\right\}\left\{\pi^+\pi^-, \pi^0\pi^0\right\}\left\{\pi^0\eta_8\right\}. \end{array}$$

(d) Also some isolated states (i.e. which do not mix) like for example $K^0\pi^-, (Y = 1, T_3 = -3/2)$.

We see from this analysis that we have to consider two-channel, three-channel and six-channel mixings under SU(3), and only two-channel mixings under SU(2).

5.1 Two-channel mixings

5.1.1 Different parametrizations of M

In this section, we will develop quite extensively the general two-channel mixing parametrizations. We start from a general orthogonal transformation (112) and M_{diag}:

$$M^{(2)} = \frac{1}{2}\begin{pmatrix} \left(1+\sqrt{1-4\varepsilon^2}\right)\mathrm{e}^{\mathrm{i}\delta_1} + \left(1-\sqrt{1-4\varepsilon^2}\right)\mathrm{e}^{\mathrm{i}\delta_2} & 2\varepsilon\left(\mathrm{e}^{\mathrm{i}\delta_2}-\mathrm{e}^{\mathrm{i}\delta_1}\right) \\ 2\varepsilon\left(\mathrm{e}^{\mathrm{i}\delta_2}-\mathrm{e}^{\mathrm{i}\delta_1}\right) & \left(1-\sqrt{1-4\varepsilon^2}\right)\mathrm{e}^{\mathrm{i}\delta_1} + \left(1+\sqrt{1-4\varepsilon^2}\right)\mathrm{e}^{\mathrm{i}\delta_2} \end{pmatrix}, \tag{56}$$

$$\begin{pmatrix} |C_1\rangle \\ |C_2\rangle \end{pmatrix} = P\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \{x_1\} \\ \{x_2\} \end{pmatrix},$$
$$M_{\mathrm{diag}} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\delta_1} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\delta_2} \end{pmatrix}. \tag{54}$$

By defining $\varepsilon = \cos\alpha\sin\alpha, \lambda = 2\varepsilon\sin\left(\frac{\delta_2-\delta_1}{2}\right)$, we obtain the following forms:

$$M^{(1)} = \begin{pmatrix} \cos^2\alpha\mathrm{e}^{\mathrm{i}\delta_1} + \sin^2\alpha\mathrm{e}^{\mathrm{i}\delta_2} & \cos\alpha\sin\alpha\left(\mathrm{e}^{\mathrm{i}\delta_2}-\mathrm{e}^{\mathrm{i}\delta_1}\right) \\ \cos\alpha\sin\alpha\left(\mathrm{e}^{\mathrm{i}\delta_2}-\mathrm{e}^{\mathrm{i}\delta_1}\right) & \sin^2\alpha\mathrm{e}^{\mathrm{i}\delta_1} + \cos^2\alpha\mathrm{e}^{\mathrm{i}\delta_2} \end{pmatrix}, \tag{55}$$

(see (56) on top of the page)

$$M^{(3)} = \begin{pmatrix} \sqrt{1-\lambda^2}\mathrm{e}^{\mathrm{i}\beta_1} & \mathrm{i}\lambda\mathrm{e}^{\mathrm{i}\left(\frac{\beta_2+\beta_1}{2}\right)} \\ \mathrm{i}\lambda\mathrm{e}^{\mathrm{i}\left(\frac{\beta_2+\beta_1}{2}\right)} & \sqrt{1-\lambda^2}\mathrm{e}^{\mathrm{i}\beta_2} \end{pmatrix}$$
$$\text{with } \begin{cases} \beta_1 = \arg\left(\cos^2\alpha\mathrm{e}^{\mathrm{i}\delta_1} + \sin^2\alpha\ \mathrm{e}^{\mathrm{i}\delta_2}\right), \\ \beta_2 = \arg\left(\sin^2\alpha\mathrm{e}^{\mathrm{i}\delta_1} + \cos^2\alpha\ \mathrm{e}^{\mathrm{i}\delta_2}\right), \end{cases} \tag{57}$$

$$M^{(4)} = \mathrm{e}^{\mathrm{i}\delta_1}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(\mathrm{e}^{\mathrm{i}(\delta_2-\delta_1)}-1\right) \times \begin{pmatrix} \sin^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \cos^2\alpha \end{pmatrix}\right]. \tag{58}$$

Here $\beta_1+\beta_2 = \delta_1+\delta_2$ and the last form is obtained from (114).

5.1.2 Parametrization of $M^2 = S$: The elasticity parameters

In this section, we will define the elasticity parameter. Let us consider a general coupled system of two states X_1 and X_2. We can describe this system in three different bases: The eigenchannel basis (C_1, C_2), with

$$S_{\mathrm{eigen}} = \begin{pmatrix} \mathrm{e}^{2\mathrm{i}\delta_1} & 0 \\ 0 & \mathrm{e}^{2\mathrm{i}\delta_2} \end{pmatrix}; \tag{59}$$

the isospin basis (T_1, T_2), with

$$S_{\mathrm{isospin}} = \begin{pmatrix} \eta_{\mathrm{w}}\mathrm{e}^{2\mathrm{i}w_1} & \mathrm{i}\sqrt{1-\eta_{\mathrm{w}}^2}\mathrm{e}^{\mathrm{i}(w_1+w_2)} \\ \mathrm{i}\sqrt{1-\eta_{\mathrm{w}}^2}\mathrm{e}^{\mathrm{i}(w_1+w_2)} & \eta_{\mathrm{w}}\mathrm{e}^{2\mathrm{i}w_2} \end{pmatrix}, \tag{60}$$

and the physical basis (X_1, X_2), with

$$S_{\mathrm{physical}} = \begin{pmatrix} \eta\mathrm{e}^{2\mathrm{i}\alpha_1} & \mathrm{i}\sqrt{1-\eta^2}\mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)} \\ \mathrm{i}\sqrt{1-\eta^2}\mathrm{e}^{\mathrm{i}(\alpha_1+\alpha_2)} & \eta\mathrm{e}^{2\mathrm{i}\alpha_2} \end{pmatrix}. \tag{61}$$

Thus, we have two different possible definitions of the elasticity parameter: (1) The parameter η_{w} quantifies the deviation of S in the isospin basis from its diagonal form in the eigenchannel basis. SU(2) elasticity implies $\eta_{\mathrm{w}} = 1$ since when $\eta_{\mathrm{w}} = 1$, $S_{\mathrm{eigen}} = S_{\mathrm{isospin}}$. The phases w_1 and w_2 are then eigenphases, sometimes called Watson phases (hence the subscript w to η_{w}).

(2) The parameter η quantifies the deviation of S in the physical basis from its diagonal form in the eigenchannel basis. This is the way the elasticity parameter will be defined in this paper. This definition allows one to define a elasticity parameter for every mixings, including SU(2) elastic mixings.

This η elasticity parameter is defined in terms of the mixing parameter in the following way (the complete discussion is in the Appendix): S_{physical} is built in the standard way, as a general coupled channel mixing:

$$S_{\mathrm{physical}}(\delta_1, \delta_2, \beta) = O^{\mathrm{t}}(\beta)S_{\mathrm{eigen}}(\delta_1, \delta_2)O(\beta), \tag{62}$$

with

$$O(\beta) = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}.$$

We can change the parameter basis from $\delta_1, \delta_2, \beta$ to α_1, α_2, η. The elasticity parameter is then given in terms of the mixing parameter as

$$\eta = \sqrt{1-4\varepsilon^2\sin^2(\delta_2-\delta_1)}; \quad \varepsilon = \cos\beta\sin\beta. \tag{63}$$

This formula is quite interesting. We can distinguish two factors contributing to η:

- The ε parameter quantifies the non-diagonal trend of $O(\beta)$, i.e. the distance between the physical basis and the eingenchannel basis.
- The elasticity parameter η quantifies the non-diagonal trend of S, which is a *function of both ε and the eigenphase difference*, since if these phases are equal, the mixings disappear.

Remarks: It is now clear that the form (60) used in some other papers to introduce inelasticity is equivalent to a general two-channel *elastic* mixing in the context of Watson's theorem. Note, however, that this form (60) is also used as a general parametrization for a 2×2 unitary symmetric matrix in the K-matrix formalism, and it no longer reduces to an elastic parametrization there.

5.1.3 SU(2 or 3) two-channel mixings in B and D decays

In this section, we will give the parameters defined above for the different two-channel mixings among two pseudoscalar states, in the framework of SU(2) and SU(3). From these parameters, one can easily rebuild the mixing matrix using one of the forms $M^{(1)}$ to $M^{(4)}$ (55–58). (see (64) on top of the next page)

Note that in all the mixings, we have η very close to $\cos(\delta_1-\delta_0)$, i.e. maximal mixings.

Flavor groups	Coupled states	Strong phases		Mixing parameters		Elasticity parameters
		δ_1	δ_2	$\cos\beta$	$\sin\beta$	
SU(3)	$\left\{K^-K^0, \pi^-\eta_8\right\}$ $\left\{K^+\overline{K^0}, \pi^+\eta_8\right\}$	δ_{27}	δ_8	$\sqrt{2/5}$	$\sqrt{3/5}$	$\varepsilon^2 = 6/25 = 0.24$ $\eta = \sqrt{1-\frac{24}{25}\sin^2(\delta_{27}-\delta_8)}$
SU(2)	$\left\{K^+\pi^-, K^0\pi^0\right\}$ $\left\{K^-\pi^+, \overline{K^0}\pi^0\right\}$	$\delta_{3/2}$	$\delta_{1/2}$	$-\sqrt{1/3}$	$\sqrt{2/3}$	$\varepsilon^2 = 2/9 \approx 0.22$ $\eta = \sqrt{1-\frac{8}{9}\sin^2(\delta_{3/2}-\delta_{1/2})}$
	$\left\{\overline{K^0}\pi^-, K^-\pi^0\right\}$ $\left\{K^0\pi^+, K^+\pi^0\right\}$	$\delta_{3/2}$	$\delta_{1/2}$	$\sqrt{1/3}$	$\sqrt{2/3}$	$\varepsilon^2 = 2/9 \approx 0.22$ $\eta = \sqrt{1-\frac{8}{9}\sin^2(\delta_{3/2}-\delta_{1/2})}$
	$\left\{\pi^+\pi^-, \pi^0\pi^0\right\}$	δ_2	δ_0	$\sqrt{1/3}$	$\sqrt{2/3}$	$\varepsilon^2 = 2/9 \approx 0.22$ $\eta = \sqrt{1-\frac{8}{9}\sin^2(\delta_2-\delta_0)}$
	$\left\{K^+K^-, K^0\overline{K}^0\right\}$	δ_1	δ_0	$\sqrt{1/2}$	$\sqrt{1/2}$	$\varepsilon^2 = 1/4$ (maximal) $\eta = \cos(\delta_1-\delta_0)$

(64)

$$\begin{pmatrix} B^+ \to K^+\overline{K^0} \\ B^+ \to \pi^+\eta_8 \end{pmatrix} = M_F^{\mathrm{SU}(3)} \begin{pmatrix} V_{ub}^*V_{ud}\,(A+P) + V_{cb}^*V_{cd}\,(P^c) + V_{tb}^*V_{td}\left(P^{\mathrm{t}}\right) \\ \frac{1}{\sqrt{6}}\left[V_{ub}^*V_{ud}\,(T+C+2A+2P) + V_{cb}^*V_{cd}\,(2P^c) + V_{tb}^*V_{td}\left(2P^{\mathrm{t}}\right)\right] \end{pmatrix}. \tag{66}$$

5.1.4 Quark diagrams

We will illustrate the application of the preceding mixing matrices on QD parametrizations of $\left\{K^-K^0, \pi^-\eta_8\right\}$ only. The other channels will be treated when dealing with three- and six-channel mixings.

The SU(3) mixings among the $E : \left\{K^-K^0, \pi^-\eta_8\right\}$ in D decays (Cabibbo approximation for CKM) is simply given by $M_E^{\mathrm{SU}(3)}$:

$$\begin{aligned} &\begin{pmatrix} D^- \to K^-K^0 \\ D^- \to \pi^-\eta_8 \end{pmatrix} \\ &= V_{cd}^*V_{ud} \begin{pmatrix} \frac{1}{5}\left(2\mathrm{e}^{\mathrm{i}\delta_{27}} + 3\mathrm{e}^{\mathrm{i}\delta_8}\right) & \frac{\sqrt{6}}{5}\left(\mathrm{e}^{\mathrm{i}\delta_8} - \mathrm{e}^{\mathrm{i}\delta_{27}}\right) \\ \frac{\sqrt{6}}{5}\left(\mathrm{e}^{\mathrm{i}\delta_8} - \mathrm{e}^{\mathrm{i}\delta_{27}}\right) & \frac{1}{5}\left(3\mathrm{e}^{\mathrm{i}\delta_{27}} + 2\mathrm{e}^{\mathrm{i}\delta_8}\right) \end{pmatrix} \\ &\quad \times \begin{pmatrix} -T + A \\ \frac{1}{\sqrt{6}}\,(T + 3C + 2A) \end{pmatrix}. \end{aligned} \tag{65}$$

We can proceed similarly in B decays (mixing $F : \left\{K^+\overline{K^0}, \pi^+\eta_8\right\}$) (see (66) on top of the page)

5.2 Three-channel mixings

5.2.1 Bare amplitudes

Consider for definiteness the set of decays $\overline{D}^0$ to $K^+\pi^-$, $K^0\pi^0$, $K^0\eta_8$. The SU(3) QD decompositions are, omitting CKM elements,

$$\begin{cases} \left(\overline{D}^0 \to \left\{K^+\pi^-\right\}\right) = \mathrm{T} + \mathrm{E}, \\ \left(\overline{D}^0 \to \left\{K^0\pi^0\right\}\right) = \frac{1}{\sqrt{2}}\,(C - E), \\ \left(\overline{D}^0 \to \left\{K^0\eta_8\right\}\right) = \frac{1}{\sqrt{6}}\,(C - E). \end{cases} \tag{67}$$

5.2.2 Physical amplitudes

We can now apply FSI mixing matrices on these bare amplitudes to obtain a parametrization of the physical decay amplitudes.

SU(3) mixings Applying the SU(3) elastic mixing matrix (10), we find the full amplitudes:

$$\begin{cases} \left(\overline{D}^0 \to K^+\pi^-\right) = T\left(\dfrac{2\mathrm{e}^{\mathrm{i}\delta_{27}} + 3\mathrm{e}^{\mathrm{i}\delta_8}}{5}\right) \\ \qquad\qquad +C\dfrac{2}{5}\left(\mathrm{e}^{\mathrm{i}\delta_{27}} - \mathrm{e}^{\mathrm{i}\delta_8}\right) + E\mathrm{e}^{\mathrm{i}\delta_8}, \\ \left(\overline{D}^0 \to K^0\pi^0\right) = \frac{1}{\sqrt{2}}\left(T\dfrac{3}{5}\left(\mathrm{e}^{\mathrm{i}\delta_{27}} - \mathrm{e}^{\mathrm{i}\delta_8}\right)\right. \\ \qquad\qquad \left. +C\left(\dfrac{3\mathrm{e}^{\mathrm{i}\delta_{27}} + 2\mathrm{e}^{\mathrm{i}\delta_8}}{5}\right) - E\mathrm{e}^{\mathrm{i}\delta_8}\right), \\ \left(\overline{D}^0 \to K^0\eta_8\right) = \frac{1}{\sqrt{6}}\left(T\dfrac{3}{5}\left(\mathrm{e}^{\mathrm{i}\delta_{27}} - \mathrm{e}^{\mathrm{i}\delta_8}\right)\right. \\ \qquad\qquad \left. +C\left(\dfrac{3\mathrm{e}^{\mathrm{i}\delta_{27}} + 2\mathrm{e}^{\mathrm{i}\delta_8}}{5}\right) - E\mathrm{e}^{\mathrm{i}\delta_8}\right). \end{cases} \tag{68}$$

$$M^{\text{general}}(\alpha)=\left(\begin{array}{c}\frac{1}{3}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\delta_{C1}}+2\mathrm{e}^{\mathrm{i}\delta_{C2}} & \sqrt{2}\left(\mathrm{e}^{\mathrm{i}\delta_{C1}}-\mathrm{e}^{\mathrm{i}\delta_{C2}}\right) & 0\\ \sqrt{2}\left(\mathrm{e}^{\mathrm{i}\delta_{C1}}-\mathrm{e}^{\mathrm{i}\delta_{C2}}\right) & 2\mathrm{e}^{\mathrm{i}\delta_{C1}}+\mathrm{e}^{\mathrm{i}\delta_{C2}} & 0\\ 0 & 0 & 3\mathrm{e}^{\mathrm{i}\delta_{C2}}\end{pmatrix}\\ +\frac{\left(\mathrm{e}^{\mathrm{i}\delta_{C3}}-\mathrm{e}^{\mathrm{i}\delta_{C2}}\right)}{3}\begin{pmatrix}2\sin^2\alpha & -\sqrt{2}\sin^2\alpha & -\sqrt{6}\cos\alpha\sin\alpha\\ -\sqrt{2}\sin^2\alpha & \sin^2\alpha & \sqrt{3}\cos\alpha\sin\alpha\\ -\sqrt{6}\cos\alpha\sin\alpha & \sqrt{3}\cos\alpha\sin\alpha & 3\cos^2\alpha\end{pmatrix}\end{array}\right). \tag{70}$$

SU(2) mixings If we choose to apply the SU(2) matrix (36), we get

$$\begin{cases}\left(\overline{D}^0\to K^+\pi^-\right)=T\left(\dfrac{\mathrm{e}^{\mathrm{i}\delta_{3/2}}+2\mathrm{e}^{\mathrm{i}\delta^{(1)}_{1/2}}}{3}\right)\\ \qquad\qquad +C\left(\dfrac{\mathrm{e}^{\mathrm{i}\delta_{3/2}}-\mathrm{e}^{\mathrm{i}\delta^{(1)}_{1/2}}}{3}\right)+E\mathrm{e}^{\mathrm{i}\delta^{(1)}_{1/2}},\\ \left(\overline{D}^0\to K^0\pi^0\right)=\frac{1}{\sqrt{2}}\left(T\left(\dfrac{2\left(\mathrm{e}^{\mathrm{i}\delta_{3/2}}-\mathrm{e}^{\mathrm{i}\delta^{(1)}_{1/2}}\right)}{3}\right)\right.\\ \qquad\qquad \left.+C\left(\dfrac{2\mathrm{e}^{\mathrm{i}\delta_{3/2}}+\mathrm{e}^{\mathrm{i}\delta^{(1)}_{1/2}}}{3}\right)-E\mathrm{e}^{\mathrm{i}\delta^{(1)}_{1/2}}\right),\\ \left(\overline{D}^0\to K^0\eta_8\right)=\frac{1}{\sqrt{6}}(C-E)\,\mathrm{e}^{\mathrm{i}\delta^{(2)}_{1/2}}.\end{cases} \tag{69}$$

General mixings Instead of applying directly the $M^{\text{general}}(\alpha)$ (50), let us rewrite it in an interesting way. Using a decomposition like (114), we write (see (70) on top of the page). We see that the first term corresponds to SU(2) mixing (see (36)), and the second one is the perturbation due to the mixing with $K^0\eta_8$. This equation shows that the mixing of $K\pi$ with $K^0\eta_8$ is a function of both the mixing parameter α and the eigenphase difference $\left(\mathrm{e}^{\mathrm{i}\delta_{C3}}-\mathrm{e}^{\mathrm{i}\delta_{C2}}\right)$ (exactly like the elasticity parameter η in two-channel mixing). If α is small, we write

$$M=\left(\begin{array}{c}\frac{1}{3}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\delta_{C1}}+2\mathrm{e}^{\mathrm{i}\delta_{C2}} & \sqrt{2}\left(\mathrm{e}^{\mathrm{i}\delta_{C1}}-\mathrm{e}^{\mathrm{i}\delta_{C2}}\right) & 0\\ \sqrt{2}\left(\mathrm{e}^{\mathrm{i}\delta_{C1}}-\mathrm{e}^{\mathrm{i}\delta_{C2}}\right) & 2\mathrm{e}^{\mathrm{i}\delta_{C1}}+\mathrm{e}^{\mathrm{i}\delta_{C2}} & 0\\ 0 & 0 & 3\mathrm{e}^{\mathrm{i}\delta_{C2}}\end{pmatrix}\\ +\frac{\left(\mathrm{e}^{\mathrm{i}\delta_{C3}}-\mathrm{e}^{\mathrm{i}\delta_{C2}}\right)}{3}\begin{pmatrix}0 & 0 & -\sqrt{6}\alpha\\ 0 & 0 & \sqrt{3}\alpha\\ -\sqrt{6}\alpha & \sqrt{3}\alpha & 3\end{pmatrix}\end{array}\right). \tag{71}$$

This form can be applied to QD parametrizations. From (67), we have for $K\pi$ decays (omitting CKM factors)

$$\begin{cases}\left(\overline{D}^0\to K^+\pi^-\right)\\ \quad=T\left(\dfrac{\mathrm{e}^{\mathrm{i}\delta_{C1}}+2\mathrm{e}^{\mathrm{i}\delta_{C2}}}{3}\right)+C\left(\dfrac{\mathrm{e}^{\mathrm{i}\delta_{C1}}-\mathrm{e}^{\mathrm{i}\delta_{C2}}}{3}\right)\\ \qquad+E\mathrm{e}^{\mathrm{i}\delta_{C2}}-\dfrac{\alpha\left(\mathrm{e}^{\mathrm{i}\delta_{C3}}-\mathrm{e}^{\mathrm{i}\delta_{C2}}\right)}{3}(C-E),\\ \left(\overline{D}^0\to K^0\pi^0\right)\\ \quad=\frac{1}{\sqrt{2}}\left(T\left(\dfrac{2\left(\mathrm{e}^{\mathrm{i}\delta_{C1}}-\mathrm{e}^{\mathrm{i}\delta_{C2}}\right)}{3}\right)+C\left(\dfrac{2\mathrm{e}^{\mathrm{i}\delta_{C1}}+\mathrm{e}^{\mathrm{i}\delta_{C2}}}{3}\right)\right.\\ \qquad\left.-E\mathrm{e}^{\mathrm{i}\delta_{C2}}+\dfrac{\alpha\left(\mathrm{e}^{\mathrm{i}\delta_{C3}}-\mathrm{e}^{\mathrm{i}\delta_{C2}}\right)}{3}(C-E),\right)\end{cases} \tag{72}$$

which is to be compared with (69).

5.2.3 Other three-channel mixings

The forms (10, 70, $\cdots$) for the mixing matrix is valid for the mixings in the sets A and D. For the mixings in B and C, the $M^{\text{SU(3)}}$ matrix is

$$M^{\text{SU(3)}}=\frac{1}{5}\begin{pmatrix}2\mathrm{e}^{\mathrm{i}\delta_{27}}+3\mathrm{e}^{\mathrm{i}\delta_8} & -\frac{3}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\delta_{27}}-\mathrm{e}^{\mathrm{i}\delta_8}\right) & \sqrt{\frac{3}{2}}\left(\mathrm{e}^{\mathrm{i}\delta_{27}}-\mathrm{e}^{\mathrm{i}\delta_8}\right)\\ -\frac{3}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\delta_{27}}-\mathrm{e}^{\mathrm{i}\delta_8}\right) & \frac{1}{2}\left(7\mathrm{e}^{\mathrm{i}\delta_{27}}+3\mathrm{e}^{\mathrm{i}\delta_8}\right) & \frac{\sqrt{3}}{2}\left(\mathrm{e}^{\mathrm{i}\delta_{27}}-\mathrm{e}^{\mathrm{i}\delta_8}\right)\\ \sqrt{\frac{3}{2}}\left(\mathrm{e}^{\mathrm{i}\delta_{27}}-\mathrm{e}^{\mathrm{i}\delta_8}\right) & \frac{\sqrt{3}}{2}\left(\mathrm{e}^{\mathrm{i}\delta_{27}}-\mathrm{e}^{\mathrm{i}\delta_8}\right) & \frac{1}{2}\left(9\mathrm{e}^{\mathrm{i}\delta_{27}}+\mathrm{e}^{\mathrm{i}\delta_8}\right)\end{pmatrix}, \tag{73}$$

which differs from (10) by some signs only. Note that these signs are not SU(3) phase-convention dependent, since phase conventions always disappear when calculating M-matrices.

5.3 Six-channel mixings

SU(2) analysis of set of states G For the last group of coupled states (G), we can make the following isospin analysis:

$$\begin{pmatrix}\left\{K\overline{K}\right\}:I=1\\ \left\{K\overline{K}\right\}:I=0\\ \left\{\eta_8\eta_8\right\}:I=0\\ \left\{\pi\pi\right\}:I=2\\ \left\{\pi\pi\right\}:I=0\\ \left\{\pi^0\eta_8\right\}:I=1\end{pmatrix}=\begin{pmatrix}-\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0\\ \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & 0\\ 0 & 0 & 0 & -\sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2}\left(e^{i\delta_0^{K\overline{K}}}+e^{i\delta_1^{K\overline{K}}}\right) & \frac{1}{2}\left(e^{i\delta_0^{K\overline{K}}}-e^{i\delta_1^{K\overline{K}}}\right) & 0 & 0 & 0 & 0 \\ \frac{1}{2}\left(e^{i\delta_0^{K\overline{K}}}-e^{i\delta_1^{K\overline{K}}}\right) & \frac{1}{2}\left(e^{i\delta_0^{K\overline{K}}}+e^{i\delta_1^{K\overline{K}}}\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\delta_0^{\eta\eta}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3}\left(e^{i\delta_2^{\pi\pi}}+2e^{i\delta_0^{\pi\pi}}\right) & \frac{\sqrt{2}}{3}\left(e^{i\delta_0^{\pi\pi}}-e^{i\delta_2^{\pi\pi}}\right) & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{3}\left(e^{i\delta_0^{\pi\pi}}-e^{i\delta_2^{\pi\pi}}\right) & \frac{1}{3}\left(2e^{i\delta_2^{\pi\pi}}+e^{i\delta_0^{\pi\pi}}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\delta_1^{\pi\eta}} \end{pmatrix}. \tag{75}$$

$$\times \begin{pmatrix} \{K^-K^+\}: \ I=1,0 \\ \left\{K^0\overline{K^0}\right\}: \ I=1,0 \\ \{\eta_8\eta_8\}: \ I=0 \\ \{\pi^+\pi^-\}: \ I=2,0 \\ \left\{\pi^0\pi^0\right\}: \ I=2,0 \\ \left\{\pi^0\eta_8\right\}: \ I=1 \end{pmatrix}. \tag{74}$$

$$\begin{pmatrix} -\frac{a}{20}-\frac{b}{5}+\frac{c}{4} & \frac{1}{\sqrt{2}}\left(-\frac{a}{20}-\frac{b}{5}+\frac{c}{4}\right) & \frac{\sqrt{3}}{5}(b-a) \\ -\frac{a}{20}-\frac{b}{5}+\frac{c}{4} & \frac{1}{\sqrt{2}}\left(-\frac{a}{20}-\frac{b}{5}+\frac{c}{4}\right) & -\frac{\sqrt{3}}{5}(b-a) \\ \frac{1}{\sqrt{2}}\left(\frac{3a}{20}-\frac{2b}{5}+\frac{c}{4}\right) & \frac{3a}{40}-\frac{b}{5}+\frac{c}{8} & 0 \\ \frac{7a}{20}+\frac{2b}{5}+\frac{c}{4} & \frac{1}{\sqrt{2}}\left(-\frac{13a}{20}+\frac{2b}{5}+\frac{c}{4}\right) & 0 \\ \frac{1}{\sqrt{2}}\left(-\frac{13a}{20}+\frac{2b}{5}+\frac{c}{4}\right) & \frac{27a}{40}+\frac{b}{5}+\frac{c}{8} & 0 \\ 0 & 0 & \frac{1}{5}(2b+3a) \end{pmatrix}, \quad \text{with} \begin{cases} a=e^{i\delta_{27}}, \\ b=e^{i\delta_8}, \\ c=e^{i\delta_1}. \end{cases} \tag{77}$$

We obtain the already introduced SU(2) mixing matrix (see Table 64): $M^{\mathrm{SU(2)}}$ which equals (see (75) on top of the page).

Most general mixing among the states of set G compatible with isospin From SU(2), to go towards SU(3), we must introduce an extra mixing between the two isospin 1 states (one mixing parameter α_1) and extra mixings between the three isospin 0 states (three mixing parameters α_2, α_3 and α_4):

$$O(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\,(\text{Intermediate states}) = \tag{76}$$

$$\begin{pmatrix} 1&0&0&0&0&0 \\ 0&x&x&0&x&0 \\ 0&x&x&0&x&0 \\ 0&0&0&1&0&0 \\ 0&x&x&0&x&0 \\ 0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} y&0&0&0&0&y \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ y&0&0&0&0&y \end{pmatrix} \begin{pmatrix} \left\{K\overline{K}\right\}: \text{ Isospin 1} \\ \left\{K\overline{K}\right\}: \text{Isospin 0} \\ \{\eta_8\eta_8\}: \text{Isospin 0} \\ \{\pi\pi\}: \text{Isospin 2} \\ \{\pi\pi\}: \text{Isospin 0} \\ \left\{\pi^0\eta_8\right\}: \text{Isospin 1} \end{pmatrix},$$

with x the entries of a 3 by 3 orthogonal matrix (three angles $\alpha_2, \alpha_3, \alpha_4$) and y the entries of a 2 by 2 orthogonal matrix (one angle α_1). By using $O(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, we can build the most general mixing $M^{\mathrm{general}}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ among this six-state set compatible with isospin.

SU(3) analysis of set of states G Finally, we can find a value for each of the four parameters $\alpha_1,\cdots,\alpha_4$ such that $M^{\mathrm{general}}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \to M^{\mathrm{SU(3)}}$ equals

$$\begin{pmatrix} \frac{7a}{20}+\frac{2b}{5}+\frac{c}{4} & -\frac{a}{20}-\frac{b}{5}+\frac{c}{4} & \frac{1}{\sqrt{2}}\left(-\frac{9a}{20}+\frac{b}{5}+\frac{c}{4}\right) \\ -\frac{a}{20}-\frac{b}{5}+\frac{c}{4} & \frac{7a}{20}+\frac{2b}{5}+\frac{c}{4} & \frac{1}{\sqrt{2}}\left(-\frac{9a}{20}+\frac{b}{5}+\frac{c}{4}\right) \\ \frac{1}{\sqrt{2}}\left(-\frac{9a}{20}+\frac{b}{5}+\frac{c}{4}\right) & \frac{1}{\sqrt{2}}\left(-\frac{9a}{20}+\frac{b}{5}+\frac{c}{4}\right) & \frac{27a}{40}+\frac{b}{5}+\frac{c}{8} \\ -\frac{a}{20}-\frac{b}{5}+\frac{c}{4} & -\frac{a}{20}-\frac{b}{5}+\frac{c}{4} & \frac{1}{\sqrt{2}}\left(\frac{3a}{20}-\frac{2b}{5}+\frac{c}{4}\right) \\ \frac{1}{\sqrt{2}}\left(-\frac{a}{20}-\frac{b}{5}+\frac{c}{4}\right) & \frac{1}{\sqrt{2}}\left(-\frac{a}{20}-\frac{b}{5}+\frac{c}{4}\right) & \frac{3a}{40}-\frac{b}{5}+\frac{c}{8} \\ \frac{\sqrt{3}}{5}(b-a) & -\frac{\sqrt{3}}{5}(b-a) & 0 \end{pmatrix} \cdots$$

SU(2) specification in the SU(3) analysis In order to introduce SU(3) breaking in the FSI, we can calculate a form like (52) for this set of states. We obtain the following matrix $M^{\mathrm{SU(2\ in\ 3)}}$ (see (78) on top of the next page) with $a = e^{i\delta_{27}^2}$, $b = e^{i\delta_{27}^1}$, $c = e^{i\delta_{27}^0}$, $d = e^{i\delta_8^1}$, $e = e^{i\delta_8^0}$, $f = e^{i\delta_1^0}$ (the notation is $\delta^{\text{isospin}}_{SU(3)\ \text{rep.}}$), where we have distinguished SU(3) phases according to isospin. The SU(3) limit can be obtained by identifying $a = b = c$, $d = e$ and the modified SU(2) limit by identifying $b = d$, $c = e = f$.

Other possibilities We can of course also limit ourself to some intermediate mixings. For example, forgetting states containing η_8, we can mix $\left\{K\overline{K}\right\}_{T=0}$ with $\{\pi\pi\}_{T=0}$ with the orthogonal matrix:

$$O^{\{K\overline{K},\pi\pi\}} = \begin{pmatrix} 1&0&0&0 \\ 0&\cos\alpha&0&-\sin\alpha \\ 0&0&1&0 \\ 0&\sin\alpha&0&\cos\alpha \end{pmatrix} \times \begin{pmatrix} -\sqrt{\frac{1}{2}}&\sqrt{\frac{1}{2}}&0&0 \\ \sqrt{\frac{1}{2}}&\sqrt{\frac{1}{2}}&0&0 \\ 0&0&-\sqrt{\frac{1}{3}}&\sqrt{\frac{2}{3}} \\ 0&0&-\sqrt{\frac{2}{3}}&-\sqrt{\frac{1}{3}} \end{pmatrix} \tag{79}$$

and build a mixing matrix $M^{\{K\overline{K},\pi\pi\}} = \left(O^{\{K\overline{K},\pi\pi\}}\right)^{\mathrm{t}} M^{\{K\overline{K},\pi\pi\}}_{\mathrm{diag}} O^{\{K\overline{K},\pi\pi\}}$, which can be used for example in D decays:

$$\begin{pmatrix}
\frac{b}{5}+\frac{3c}{20}+\frac{3d}{10}+\frac{e}{10}+\frac{f}{4} & -\frac{b}{5}+\frac{3c}{20}-\frac{3d}{10}+\frac{e}{10}+\frac{f}{4} & \frac{1}{\sqrt{2}}\left(-\frac{9c}{20}+\frac{e}{5}+\frac{f}{4}\right) \\
-\frac{b}{5}+\frac{3c}{20}-\frac{3d}{10}+\frac{e}{10}+\frac{f}{4} & \frac{b}{5}+\frac{3c}{20}+\frac{3d}{10}+\frac{e}{10}+\frac{f}{4} & \frac{1}{\sqrt{2}}\left(-\frac{9c}{20}+\frac{e}{5}+\frac{f}{4}\right) \\
\frac{1}{\sqrt{2}}\left(-\frac{9c}{20}+\frac{e}{5}+\frac{f}{4}\right) & \frac{1}{\sqrt{2}}\left(-\frac{9c}{20}+\frac{e}{5}+\frac{f}{4}\right) & \frac{27c}{40}+\frac{e}{5}+\frac{f}{8} \\
-\frac{c}{20}-\frac{e}{5}+\frac{f}{4} & -\frac{c}{20}-\frac{e}{5}+\frac{f}{4} & \frac{1}{\sqrt{2}}\left(\frac{3c}{20}-\frac{2e}{5}+\frac{f}{4}\right) \\
\frac{1}{\sqrt{2}}\left(-\frac{c}{20}-\frac{e}{5}+\frac{f}{4}\right) & \frac{1}{\sqrt{2}}\left(-\frac{c}{20}-\frac{e}{5}+\frac{f}{4}\right) & \frac{3c}{40}-\frac{e}{5}+\frac{f}{8} \\
\frac{\sqrt{3}}{5}(d-b) & -\frac{\sqrt{3}}{5}(d-b) & 0
\end{pmatrix} \cdots$$

$$\cdots \begin{pmatrix}
-\frac{c}{20}-\frac{e}{5}+\frac{f}{4} & \frac{1}{\sqrt{2}}\left(-\frac{c}{20}-\frac{e}{5}+\frac{f}{4}\right) & \frac{\sqrt{3}}{5}(d-b) \\
-\frac{c}{20}-\frac{e}{5}+\frac{f}{4} & \frac{1}{\sqrt{2}}\left(-\frac{c}{20}-\frac{e}{5}+\frac{f}{4}\right) & -\frac{\sqrt{3}}{5}(d-b) \\
\frac{1}{\sqrt{2}}\left(\frac{3c}{20}-\frac{2e}{5}+\frac{f}{4}\right) & \frac{3c}{40}-\frac{e}{5}+\frac{f}{8} & 0 \\
\frac{a}{3}+\frac{c}{60}+\frac{2e}{5}+\frac{f}{4} & \frac{1}{\sqrt{2}}\left(-\frac{2a}{3}+\frac{c}{60}+\frac{2e}{5}+\frac{f}{4}\right) & 0 \\
\frac{1}{\sqrt{2}}\left(-\frac{2a}{3}+\frac{c}{60}+\frac{2e}{5}+\frac{f}{4}\right) & \frac{2a}{3}+\frac{c}{120}+\frac{e}{5}+\frac{f}{8} & 0 \\
0 & 0 & \frac{1}{5}(2d+3b)\,,
\end{pmatrix} \tag{78}$$

$$\begin{pmatrix} \overline{D^0}\to K^+K^- \\ \overline{D^0}\to K^0\overline{K^0} \\ \overline{D^0}\to \pi^+\pi^- \\ \overline{D^0}\to \pi^0\pi^0 \end{pmatrix} = M^{\{K\overline{K},\pi\pi\}} \begin{pmatrix} \overline{D^0}\to \{K^+K^-\} \\ \overline{D^0}\to \left\{K^0\overline{K^0}\right\} \\ \overline{D^0}\to \{\pi^+\pi^-\} \\ \overline{D^0}\to \{\pi^0\pi^0\} \end{pmatrix}. \tag{80}$$

5.3.1 Application to QD decompositions

All these matrices can now be applied on quark diagram decompositions. For D decays, using a Cabibbo approximation (and omitting CKM):

$$\begin{pmatrix} \overline{D^0}\to K^+K^- \\ \overline{D^0}\to K^0\overline{K^0} \\ \overline{D^0}\to \eta_8\eta_8 \\ \overline{D^0}\to \pi^+\pi^- \\ \overline{D^0}\to \pi^0\pi^0 \\ \overline{D^0}\to \pi^0\eta_8 \end{pmatrix} = M \begin{pmatrix} \overline{D^0}\to \{K^+K^-\} = -T-E \\ \overline{D^0}\to \left\{K^0\overline{K^0}\right\} = 0 \\ \overline{D^0}\to \{\eta_8\eta_8\} = \frac{1}{\sqrt{2}}(C-E) \\ \overline{D^0}\to \{\pi^+\pi^-\} = T+E \\ \overline{D^0}\to \left\{\pi^0\pi^0\right\} = \frac{1}{\sqrt{2}}(-C+E) \\ \overline{D^0}\to \left\{\pi^0\eta_8\right\} = \frac{1}{\sqrt{3}}(C-E) \end{pmatrix}. \tag{81}$$

For B^0 and B_s decays into these channels, the QD parametrizations of decay amplitudes are given in the appendix.

Example: D decays to $K\overline{K}$. As is well-known, the amplitudes for the decay $\overline{D^0}\to K^0\overline{K^0}$ is identically zero under SU(3). Here we can see that this decay is zero at the level of bare amplitude. Of course, if we apply the SU(3) FSI matrix, the full amplitude remains zero. Under SU(2), we find as usual

$$\begin{pmatrix} \overline{D^0}\to K^+K^- \\ \overline{D^0}\to K^0\overline{K^0} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\delta_0^{K\overline{K}}}+\mathrm{e}^{\mathrm{i}\delta_1^{K\overline{K}}}\right)(T+E) \\ -\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\delta_0^{K\overline{K}}}-\mathrm{e}^{\mathrm{i}\delta_1^{K\overline{K}}}\right)(T+E) \end{pmatrix}. \tag{82}$$

If we use the $M^{\mathrm{SU(2\ in\ 3)}}$ form (78), we find

$$\begin{cases} \overline{D^0}\to K^+K^- = \\ -T\left(\dfrac{2(\mathrm{e}^{\mathrm{i}\delta_{27}^0}+\mathrm{e}^{\mathrm{i}\delta_{27}^1})+3(\mathrm{e}^{\mathrm{i}\delta_8^0}+\mathrm{e}^{\mathrm{i}\delta_8^1})}{10}\right) \\ +C\left(\dfrac{(\mathrm{e}^{\mathrm{i}\delta_8^0}+\mathrm{e}^{\mathrm{i}\delta_8^1})-(\mathrm{e}^{\mathrm{i}\delta_{27}^0}+\mathrm{e}^{\mathrm{i}\delta_{27}^1})}{5}\right)-E\left(\dfrac{\mathrm{e}^{\mathrm{i}\delta_8^0}+\mathrm{e}^{\mathrm{i}\delta_8^1}}{2}\right), \\ \overline{D^0}\to K^0\overline{K^0} = \\ -T\left(\dfrac{2(\mathrm{e}^{\mathrm{i}\delta_{27}^0}-\mathrm{e}^{\mathrm{i}\delta_{27}^1})+3(\mathrm{e}^{\mathrm{i}\delta_8^0}-\mathrm{e}^{\mathrm{i}\delta_8^1})}{10}\right) \\ +C\left(\dfrac{(\mathrm{e}^{\mathrm{i}\delta_8^0}-\mathrm{e}^{\mathrm{i}\delta_8^1})-(\mathrm{e}^{\mathrm{i}\delta_{27}^0}-\mathrm{e}^{\mathrm{i}\delta_{27}^1})}{5}\right)-E\left(\dfrac{\mathrm{e}^{\mathrm{i}\delta_8^0}-\mathrm{e}^{\mathrm{i}\delta_8^1}}{2}\right). \end{cases} \tag{83}$$

Here we can see that the non-zero $\overline{D^0}\to K^0\overline{K^0}$ is generated by SU(3) breaking in the FSI phases. If we identify $\delta_{27}^T=\delta_8^T=\delta_T^{K\overline{K}}$, we recover the SU(2) limit (82) and if we identify $\delta_R^1=\delta_R^0=\delta_R$, we recover the SU(3) limit $\left(\overline{D^0}\to K^0\overline{K^0}\right)=0$. Experimentally, the amplitude for $\overline{D^0}\to K^0\overline{K^0}$ is non-negligible compared to $\overline{D^0}\to K^+K^-$; the SU(3) breaking is therefore quite important. Note that this interpretation of the non-zero amplitude $\left(\overline{D^0}\to K^0\overline{K^0}\right)$ given here is not new, but it shows the simplicity of the proposed matrix method.

6 Conclusion

The main motivation of our work is to obtain a parametrization of B and D decays which can be used to extract some theoretically interesting quantities from experimental measurements.

The first step towards this parametrization is the generalized Watson theorem $W=S^{1/2}W_{\mathrm{b}}$ applied to a set

of decay channels. This theorem shows that physical decay amplitudes can be factorized into a bare part and an FSI part. The model character of our procedure enters precisely when identifying those bare amplitudes to elementary processes free of FSI effect, this identification being strictly equivalent to the elastic hypothesis (as soon as the **S**-matrix is unitary). Elementary processes and FSI effects can then be analysed separately. Quark diagrams are used at the bare level, and this ensures that they are well-defined in terms of elementary processes. For the FSI part, we introduce unitary mixing matrices. Unitarity of these mixing matrices is equivalent to probability conservation among the set of decay channels, i.e. to elasticity. The important point is that FSIs are treated at the hadronic level, since our **S**-matrix was built from hadron states. We have then shown how to build mixing matrices, using a symmetry group or introducing selected mixings among the hadron states.

The next step should be to simplify the parametrizations obtained. Indeed, if we introduce FSI as some general mixings, we introduce many mixing parameters and many strong phases, and since we have only a limited number of possible decays, we have to reduce the number of parameters. Dynamical considerations can lead to the neglect of some quark diagrams (usually, A and PA), and also to the neglect of some mixings among the possible final states. As we have repeatedly emphasized, it is also possible to use flavor symmetry to fix some mixings. Finally, Regge phenomenology may be useful to calculate some strong phases.

The final step is of course comparison with experimental data. Within our model framework, one can build simple parametrizations. Whether the various hypotheses, emphasized in this work, are valid or not can then be tested, especially the elasticity of FSI, and the limited extend of the set of rescattering channels. For example, in B decays, treating FSI as elastic under SU(2) may be a sufficient approximation. But it could also happen that mixing with η, η' or charmed meson states are important (the present approach is straightforwardly extended to these mixings), or even mixings with multibody sates. Finally, the present elastic approach for FSI could be inappropriate. However, in that case, it will have shown where and how severely inelasticity comes into play.

In our model framework, all these considerations are possible because of the factorization of physical amplitudes and because of the identification of quark diagrams with elementary processes. In conclusion, the framework we propose may lead to a simplified parametrization (using relevant symmetry and dynamical arguments) that can be used to analyze experimental data.

Acknowledgements. Many thanks are due to J.-M. Gérard, J. Pestieau and J. Weyers for the numerous discussions, comments and encouragements. This work is supported in part by the Fonds National de la Recherche Scientifique (Belgique).

Appendices

1 SU(3) analysis of B and D decay amplitudes

Let us describe briefly the SU(3) analysis of decay amplitudes (see [1,2]). We work with the conventions that (u,d,s) transform as 3 and $(\overline{s},\overline{d},-\overline{u})$ transform as $\overline{3}$.

1.1 Initial states

For the decaying mesons, we are considering $\left(B^+,B^0,B_s\right)$ and $(\overline{D}^0,D^-,D_s^-)$ because they both transform as 3 under SU(3).

1.2 Weak hamiltonians

We give here the weak hamiltonians at lowest order in the electroweak interaction for $\Delta C=0$ B decays and for D decays. These hamiltonians are written as a Fermi current–current interaction, and the V–A structure is omitted.

(i) For B decays The weak hamiltonian for ΔC=0 transitions is at the lowest order

$$\begin{cases} H_{\rm W}^{\Delta S=0} = V_{ub}^* V_{ud} \overline{b}u.\overline{u}d + V_{cb}^* V_{cd} \overline{b}c.\overline{c}d + V_{tb}^* V_{td} \overline{b}t.\overline{t}d, \\ H_{\rm W}^{\Delta S=1} = V_{ub}^* V_{us} \overline{b}u.\overline{u}s + V_{cb}^* V_{cs} \overline{b}c.\overline{c}s + V_{tb}^* V_{ts} \overline{b}t.\overline{t}s. \end{cases} \tag{84}$$

This is written in terms of SU(3) representations as

$$H_{\rm W}^{\Delta S=0} = \begin{cases} V_{ub}^* V_{ud} \left(\sqrt{8}\left|\overline{15},\frac{3}{2},\frac{1}{2},-\frac{1}{3}\right\rangle + \left|\overline{15},\frac{1}{2},\frac{1}{2},-\frac{1}{3}\right\rangle \right. \\ \left. + \left|6,\frac{1}{2},\frac{1}{2},-\frac{1}{3}\right\rangle + \left|\overline{3},\frac{1}{2},\frac{1}{2},-\frac{1}{3}\right\rangle\right) \\ V_{cb}^* V_{cd} \left|\overline{3_c},\frac{1}{2},\frac{1}{2},-\frac{1}{3}\right\rangle + V_{tb}^* V_{td} \left|\overline{3_t},\frac{1}{2},\frac{1}{2},-\frac{1}{3}\right\rangle, \end{cases}$$

$$H_{\rm W}^{\Delta S=1} = \begin{cases} +V_{ub}^* V_{us} \left(\sqrt{6}\left|\overline{15},1,0,\frac{2}{3}\right\rangle + \sqrt{3}\left|\overline{15},0,0,\frac{2}{3}\right\rangle \right. \\ \left. - \left|6,1,0,\frac{2}{3}\right\rangle + \left|\overline{3},0,0,\frac{2}{3}\right\rangle\right) \\ +V_{cb}^* V_{cs} \left|\overline{3_c},0,0,\frac{2}{3}\right\rangle + V_{tb}^* V_{ts} \left|\overline{3_t},0,0,\frac{2}{3}\right\rangle. \end{cases} \tag{85}$$

(ii) For D decays We have

$$\begin{cases} H_{\rm W}^{\Delta S=-1} = V_{cd}^* V_{us} \overline{c}d.\overline{s}u, \\ H_{\rm W}^{\Delta S=0} = V_{cd}^* V_{ud} \overline{c}d.\overline{d}u + V_{cs}^* V_{us} \overline{c}s.\overline{s}u + V_{cb}^* V_{ub} \overline{c}b.\overline{b}u, \\ H_{\rm W}^{\Delta S=+1} = V_{cs}^* V_{ud} \overline{c}s.\overline{d}u, \end{cases} \tag{86}$$

and in representations

$$H_{\rm W}^{\Delta S=-1} = V_{cd}^* V_{us} \left(-\sqrt{12}\left|\overline{15},1,0,-\frac{4}{3}\right\rangle + \sqrt{2}\left|6,0,0,-\frac{4}{3}\right\rangle\right),$$

$$H_{\rm W}^{\Delta S=0} = \begin{cases} V_{cd}^* V_{ud} \left(-\sqrt{8}\left|\overline{15},\frac{3}{2},\frac{-1}{2},\frac{-1}{3}\right\rangle + \left|\overline{15},\frac{1}{2},\frac{-1}{2},\frac{-1}{3}\right\rangle \right. \\ \left. + \left|6,\frac{1}{2},\frac{-1}{2},\frac{-1}{3}\right\rangle + \left|\overline{3},\frac{1}{2},\frac{-1}{2},\frac{-1}{3}\right\rangle\right) \\ +V_{cs}^* V_{us} \left(-\sqrt{9}\left|\overline{15},\frac{1}{2},\frac{-1}{2},\frac{-1}{3}\right\rangle - \left|6,\frac{1}{2},\frac{-1}{2},\frac{-1}{3}\right\rangle \right. \\ \left. + \left|\overline{3},\frac{1}{2},\frac{-1}{2},\frac{-1}{3}\right\rangle\right) + V_{cb}^* V_{ub} \left|\overline{3_b},\frac{1}{2},\frac{-1}{2},\frac{-1}{3}\right\rangle, \end{cases}$$

$$H_{\rm W}^{\Delta S=+1} = V_{cs}^* V_{ud} \left(-\sqrt{12}\left|\overline{15},1,-1,\frac{2}{3}\right\rangle - \sqrt{2}\left|6,1,-1,\frac{2}{3}\right\rangle\right). \tag{87}$$

With the Cabibbo approximation ($V_{cd}^* V_{ud} = -V_{cs}^* V_{us} = \lambda, V_{cb}^* V_{ub} = 0$):

$$H_{\rm W}^{\Delta S=0} = \lambda\Bigg(-\sqrt{8}\left|\overline{15},\frac{3}{2},\frac{-1}{2},\frac{-1}{3}\right\rangle + 4\left|\overline{15},\frac{1}{2},\frac{-1}{2},\frac{-1}{3}\right\rangle + 2\left|6,\frac{1}{2},\frac{-1}{2},\frac{-1}{3}\right\rangle\Bigg). \tag{88}$$

1.3 Final states

The pseudoscalars transform as the octet 8; therefore all the final states of two charmless pseudoscalars can be obtained from the symmetric part (Bose statistics) of the tensor product $(8\otimes 8)_{\rm S} = 27, 8_{\rm S}, 1$. Note that under our conventions, the pseudoscalar octet is

$$\left(K^+, K^0, \pi^+, -\pi^0, -\pi^-, -\eta_8, \overline{K^0}, -K^-\right). \tag{89}$$

1.4 Decay amplitudes

The complete set of decays can be parametrized with SU(3) amplitudes using the standard Wigner–Eckart theorem. The results are well known (see [2–11]), but we give them again for convenience, and because we have renormalized the SU(3) amplitudes coherently in B and D decays. This implies that we have a nice correspondence between SU(3) amplitudes and quark diagrams, valid in both B and D decays. The following table summarizes the notations for the matrix elements (see (90) on top of the next page) Of course, SU(3) amplitudes have different values in B and D decays, but the SU(3) structure is similar. For D decays, we can use the Cabibbo approximation for CKM, under which only three SU(3) amplitudes survive: A^{27}, A^8 and B^8.

Remark: The procedure just described can be applied to find SU(3) decompositions of bare or of full decay amplitudes in terms of bare or full SU(3) amplitudes respectively, since FSIs proceed only by strong interactions (to translate from full to bare, just replace A^{27} by $A_{\rm b}^{27}$ and so on, see (6) and (8)). If we work at the level of full amplitudes under SU(3), this means that we are imposing SU(3) invariance on FSI.

2 Quark diagram analysis

The decompositions of bare decay amplitudes in terms of QDs are calculated as usual:

(i) For a given initial state, and a given type of QD, write all the possible flavor "final" states (note that since we are working at the bare level, "final" means here intermediate).

SU(3) Amplitudes	$3 \otimes \overline{15} \to 8, 27$	$3 \otimes 6 \to 8$	$3 \otimes \overline{3} \to 1, 8$	$3 \otimes \overline{3_q} \to 1, 8$
Initial = 3	$A^{27} = \left\langle 27 \left\vert \overline{15} \right\vert 3 \right\rangle$	$B^8 = \langle 8 \vert 6 \vert 3 \rangle$	$C^8 = \left\langle 8 \left\vert \overline{3} \right\vert 3 \right\rangle$	$C^{8q} = \left\langle 8 \left\vert \overline{3_q} \right\vert 3 \right\rangle$
Final = 27,8,1	$A^8 = \left\langle 8 \left\vert \overline{15} \right\vert 3 \right\rangle$		$C^1 = \left\langle 1 \left\vert \overline{3} \right\vert 3 \right\rangle$	$C^{1q} = \left\langle 1 \left\vert \overline{3_q} \right\vert 3 \right\rangle$

(90)

It is at this step that we ensure SU(3) symmetry. By identifying diagrams which correspond under the exchange of u, d or s (and $\overline{u}$, $\overline{d}$, $\overline{s}$), we are left with the six diagrams T, C, E, A, P, PA, and some P^q and PA^q (proceeding via a heavy quark q in the loop). We can also implement SU(2) by considering exchange of u, d (and $\overline{u}$, $\overline{d}$), but we are left with a huge number of different diagrams.
(ii) Contract these flavor "final" states with every hadron state according to the conventions

$$\begin{array}{lll} K^+ = u\overline{s}, & \overline{K^0} = s\overline{u}, & \pi^+ = u\overline{d}, \\ K^0 = d\overline{s}, & K^- = s\overline{d}, & \pi^- = d\overline{u}, \end{array} \quad (91)$$

$$\pi^0 = \tfrac{1}{\sqrt{2}}\left(u\overline{u} - d\overline{d}\right), \qquad \eta_8 = \tfrac{1}{\sqrt{6}}\left(u\overline{u} + d\overline{d} - 2s\overline{s}\right).$$

This is necessary in order to render QD decompositions compatible with SU(3) decompositions (with the phase conventions (89)).
(ii) If the "final" state contains identical hadrons, divide by $2^{1/2}$. This is compulsory in order to compare with SU(3) amplitudes where final states are symmetric under exchange (Bose statistics). This implies that when we are calculating decay widths for identical-particle "final" states, we should not divide by 2.
(iv) Finally, add the required CKM elements.

Note that even if the analysis is the same in B and D, the specific values of the QDs are of course different for B and D decays.

2.1 Link between SU(3) bare amplitudes and QD amplitudes

Since we have two parametrizations: QD and SU(3) amplitudes (bare), we can find relations between them. The expressions of SU(3) amplitudes in terms of QDs are

$$\begin{cases} A_{\rm b}^{27} = -\frac{1}{10}\left(T + C\right), \\ A_{\rm b}^{8} = \frac{1}{40}\left(T + C\right) + \frac{1}{8}\left(E + A\right), \\ B_{\rm b}^{8} = \frac{1}{4}\left(-T + C - E + A\right), \\ C_{\rm b}^{8} = \frac{1}{8}\left(-3T + C + E - 3A\right) - P, \\ C_{\rm b}^{1} = \frac{1}{12}\left(3T - C\right) + \frac{2}{3}\left(E + P\right) + PA. \end{cases} \quad (92)$$

These relations are valid for B and D decays. There are also relations linking $\overline{3^q}$ with P^q and PA^q ($q = c, b$ or t):

$$\begin{cases} C_{\rm b}^{8q} = -P^q, \\ C_{\rm b}^{1q} = \frac{2}{3}P^q + PA^q. \end{cases} \quad (93)$$

Since there are more QD amplitudes than SU(3) amplitudes, there is a combination of QDs that never appear in decay amplitudes. This relation is

$$T - C - E + A - P + PA = 0. \quad (94)$$

This relation is to be interpreted as a relation for the corresponding coefficients in decay amplitude decompositions. For example,

$$B^+ \to \left\{K^+\pi^0\right\} = \frac{1}{\sqrt{2}}\left(T + C + A + P\right),$$

and the relation is verified:

$$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0.$$

Due to this relation, the expressions of QDs in terms of SU(3) amplitudes are not uniquely defined. Anyway, one can use the following simple set of relations to translate QDs decompositions into SU(3) amplitude decompositions:

$$\begin{cases} T = -6A_{\rm b}^{27}, \\ C = -4A_{\rm b}^{27}, \\ E = 2A_{\rm b}^{27} + 4A_{\rm b}^{8} - 2B_{b}^{8}, \\ A = 4A_{\rm b}^{8} + 2B_{b}^{8}, \\ P = 2A_{\rm b}^{27} - A_{\rm b}^{8} - B_{b}^{8} - C_{b}^{8}, \\ PA = -\frac{3}{2}A_{\rm b}^{27} - 2A_{\rm b}^{8} + 2B_{\rm b}^{8} + \frac{2}{3}C_{b}^{8} + C_{\rm b}^{1}; \end{cases} \quad (95)$$

$$\begin{cases} P^q = -C_{\rm b}^{8q}, \\ PA^q = \frac{2}{3}C_{\rm b}^{8q} + C_{\rm b}^{1q}. \end{cases} \quad (96)$$

However, it should be clear that care is needed when dealing with these relations.

3 B decays decompositions

$\Delta S = 0$ $B \to PP$		$Prefactors$	$V^*_{ub}V_{ud}$			$V^*_{ub}V_{ud}C^8$ $+V^*_{cb}V_{cd}C^{8c}$ $+V^*_{tb}V_{td}C^{8t}$	$V^*_{ub}V_{ud}C^1$ $+V^*_{cb}V_{cd}C^{1c}$ $+V^*_{tb}V_{td}C^{1t}$
			A^{27}	A^8	B^8		
B_S	$\overline{K^0}\eta_8$	$\frac{1}{\sqrt{6}}$	-6	1	1	1	0
	$\overline{K^0}\pi^0$	$\frac{1}{\sqrt{2}}$	-6	1	1	1	0
	$K^-\pi^+$	1	-4	-1	-1	-1	0
B^+	$\pi^+\eta_8$	$\frac{1}{\sqrt{6}}$	-6	6	2	-2	0
	$\overline{K^0}K^+$	1	2	3	1	-1	0
	$\pi^+\pi^0$	$\frac{1}{\sqrt{2}}$	-10	0	0	0	0
B^0	K^+K^-	1	1/2	2	0	2/3	1
	$K^0\overline{K^0}$	1	1/2	-3	1	$-1/3$	1
	$\eta_8\eta_8$	$\frac{1}{\sqrt{2}}$	$-3/2$	-1	1	1/3	1
	$\pi^0\eta_8$	$\frac{1}{\sqrt{3}}$	0	5	-1	1	0
	$\pi^+\pi^-$	1	$-7/2$	1	-1	$-1/3$	1
	$\pi^0\pi^0$	$\frac{1}{\sqrt{2}}$	13/2	1	-1	$-1/3$	1

$\Delta S = 1$ $B \to PP$		$Prefactors$	$V^*_{ub}V_{us}$			$V^*_{ub}V_{us}C^8$ $+V^*_{cb}V_{cs}C^{8c}$ $+V^*_{tb}V_{ts}C^{8t}$	$V^*_{ub}V_{us}C^1$ $+V^*_{cb}V_{cs}C^{1c}$ $+V^*_{tb}V_{ts}C^{1t}$
			A^{27}	A^8	B^8		
B^0	$K^0\eta_8$	$\frac{1}{\sqrt{6}}$	-6	1	1	1	0
	$K^0\pi^0$	$\frac{1}{\sqrt{2}}$	-6	1	1	1	0
	$K^+\pi^-$	1	-4	-1	-1	-1	0
B^+	$K^+\eta_8$	$\frac{1}{\sqrt{6}}$	-12	-3	-1	1	0
	$K^+\pi^0$	$\frac{1}{\sqrt{2}}$	-8	3	1	-1	0
	$K^0\pi^+$	1	2	3	1	-1	0
B_S	K^+K^-	1	$-7/2$	1	-1	$-1/3$	1
	$K^0\overline{K^0}$	1	1/2	-3	1	$-1/3$	1
	$\eta_8\eta_8$	$\frac{1}{\sqrt{2}}$	9/2	-2	0	$-2/3$	1
	$\pi^0\eta_8$	$\frac{1}{\sqrt{3}}$	6	4	-2	0	0
	$\pi^+\pi^-$	1	1/2	2	0	2/3	1
	$\pi^0\pi^0$	$\frac{1}{\sqrt{2}}$	1/2	2	0	2/3	1

$\Delta S = 0$ $B \to PP$		$Prefactors$	$V^*_{ub}V_{ud}$				$V^*_{ub}V_{ud}P$ $+V^*_{cb}V_{cd}P^c$ $+V^*_{tb}V_{td}P^t$	$V^*_{ub}V_{ud}PA$ $+V^*_{cb}V_{cd}PA^c$ $+V^*_{tb}V_{td}PA^t$
			T	C	E	A		
B_S	$\overline{K^0}\eta_8$	$\frac{1}{\sqrt{6}}$	0	1	0	0	-1	0
	$\overline{K^0}\pi^0$	$\frac{1}{\sqrt{2}}$	0	1	0	0	-1	0
	$K^-\pi^+$	1	1	0	0	0	1	0
B^+	$\pi^+\eta_8$	$\frac{1}{\sqrt{6}}$	1	1	0	2	2	0
	$\overline{K^0}K^+$	1	0	0	0	1	1	0
	$\pi^+\pi^0$	$\frac{1}{\sqrt{2}}$	1	1	0	0	0	0
B^0	K^+K^-	1	0	0	1	0	0	1
	$K^0\overline{K^0}$	1	0	0	0	0	1	1
	$\eta_8\eta_8$	$\frac{1}{\sqrt{2}}$	0	1/3	1/3	0	1/3	1
	$\pi^0\eta_8$	$\frac{1}{\sqrt{3}}$	0	0	1	0	-1	0
	$\pi^+\pi^-$	1	1	0	1	0	1	1
	$\pi^0\pi^0$	$\frac{1}{\sqrt{2}}$	0	-1	1	0	1	1

$\Delta S = 1$ $B \to PP$		$Prefactors$	$V^*_{ub}V_{us}$				$V^*_{ub}V_{us}P$ $+V^*_{ub}V_{us}P^c$ $+V^*_{tb}V_{ts}P^{\rm t}$	$V^*_{ub}V_{us}PA$ $+V^*_{cb}V_{cs}PA^c$ $+V^*_{tb}V_{ts}PA^{\rm t}$
			T	C	E	A		
B^0	$K^0\eta_8$	$\frac{1}{\sqrt{6}}$	0	1	0	0	-1	0
	$K^0\pi^0$	$\frac{1}{\sqrt{2}}$	0	1	0	0	-1	0
	$K^+\pi^-$	1	1	0	0	0	1	0
B^+	$K^+\eta_8$	$\frac{1}{\sqrt{6}}$	1	1	0	-1	-1	0
	$K^+\pi^0$	$\frac{1}{\sqrt{2}}$	1	1	0	1	1	0
	$K^0\pi^+$	1	0	0	0	1	1	0
B_S	K^+K^-	1	1	0	1	0	1	1
	$K^0\overline{K^0}$	1	0	0	0	0	1	1
	$\eta_8\eta_8$	$\frac{1}{\sqrt{2}}$	0	$-2/3$	$1/3$	0	$4/3$	1
	$\pi^0\eta_8$	$\frac{1}{\sqrt{3}}$	0	-1	1	0	0	0
	$\pi^+\pi^-$	1	0	0	1	0	0	1
	$\pi^0\pi^0$	$\frac{1}{\sqrt{2}}$	0	0	1	0	0	1

4 D decays decompositions

(Cabibbo approximation)

$\Delta S = 0$ $\overline{D} \to PP$		$Prefactors$	λ			λ			
			A^{27}	A^8	B^8	T	C	E	A
D^-	K^0K^-	1	6	4	2	-1	0	0	1
	$\pi^-\eta_8$	$\frac{1}{\sqrt{6}}$	-18	8	4	1	3	0	2
	$\pi^-\pi^0$	$\frac{1}{\sqrt{2}}$	10	0	0	-1	-1	0	0
D_S^-	$K^-\eta_8$	$\frac{1}{\sqrt{6}}$	-24	4	2	2	3	0	1
	$K^-\pi^0$	$\frac{1}{\sqrt{2}}$	4	-4	-2	0	-1	0	-1
	$\overline{K^0}\pi^-$	1	-6	-4	-2	1	0	0	-1
$\overline{D^0}$	K^+K^-	1	4	-4	2	-1	0	-1	0
	$K^0\overline{K^0}$	1	0	0	0	0	0	0	0
	$\eta_8\eta_8$	$\frac{1}{\sqrt{2}}$	-6	-4	2	0	1	-1	0
	$\pi^0\eta_8$	$\frac{1}{\sqrt{3}}$	-6	-4	2	0	1	-1	0
	$\pi^+\pi^-$	1	-4	4	-2	1	0	1	0
	$\pi^0\pi^0$	$\frac{1}{\sqrt{2}}$	6	4	-2	0	-1	1	0

$\Delta S = -1$ $\overline{D} \to PP$		$Prefactors$	$V^*_{cd}V_{us}$			$V^*_{cd}V_{us}$			
			A^{27}	A^8	B^8	T	C	E	A
D^-	$K^-\eta_8$	$\frac{1}{\sqrt{6}}$	-6	-4	-2	1	0	0	-1
	$K^-\pi^0$	$\frac{1}{\sqrt{2}}$	6	4	2	-1	0	0	1
	$\overline{K^0}\pi^-$	1	-4	4	2	0	1	0	1
D_S^-	$\overline{K^0}K^-$	1	-10	0	0	1	1	0	0
$\overline{D^0}$	$\overline{K^0}\eta_8$	$\frac{1}{\sqrt{6}}$	-6	-4	2	0	1	-1	0
	$\overline{K^0}\pi^0$	$\frac{1}{\sqrt{2}}$	-6	-4	2	0	1	-1	0
	$K^-\pi^+$	1	-4	4	-2	1	0	1	0

$\Delta S=+1$ $\overline{D}\to PP$		$Prefactors$	$V_{cs}^*V_{ud}$			$V_{cs}^*V_{ud}$			
			A^{27}	A^8	B^8	T	C	E	A
D_S^-	K^0K^-	1	-4	4	2	0	1	0	1
	$\pi^-\eta_8$	$\frac{1}{\sqrt{6}}$	12	8	4	-2	0	0	2
	$\pi^-\pi^0$	$\frac{1}{\sqrt{2}}$	0	0	0	0	0	0	0
D^-	$K^0\pi^-$	1	-10	0	0	1	1	0	0
$\overline{D^0}$	$K^0\eta_8$	$\frac{1}{\sqrt{6}}$	-6	-4	2	0	1	-1	0
	$K^0\pi^0$	$\frac{1}{\sqrt{2}}$	-6	-4	2	0	1	-1	0
	$K^+\pi^-$	1	-4	4	-2	1	0	1	0

5 Demonstration of the generalized Watson theorem

The **S**-matrix is given by

$$\begin{pmatrix} 1 & \mathrm{i}W_1^{\mathrm{t}} \\ \mathrm{i}CP(W_1)\equiv \mathrm{i}W_2 & S \end{pmatrix}. \tag{97}$$

Unitarity implies, in the lowest order in electroweak interactions that

$$\begin{aligned} \mathbf{S}^\dagger\mathbf{S}=\mathbf{S}\mathbf{S}^\dagger=1 &\Longleftrightarrow \begin{cases} S^\dagger S=SS^\dagger=1 \\ W_1=SW_2^* \\ W_2=SW_1^* \end{cases} \\ &\Longleftrightarrow \begin{cases} S \quad \text{unitary}, \\ (W_1+W_2)=S\ (W_1+W_2)^*, \\ (W_1-W_2)=-S\ (W_1-W_2)^*. \end{cases} \end{aligned} \tag{98}$$

Since S is symmetric and unitary, there is a real orthogonal transformation which diagonalizes it:

$$S=O^{\mathrm{t}}S_{\mathrm{diag}}O, \tag{99}$$

with, since S is unitary, a diagonal form like

$$S_{\mathrm{diag}}=\begin{pmatrix} \mathrm{e}^{2\mathrm{i}\delta_1} & 0 & \cdots 0 \\ 0 & \mathrm{e}^{2\mathrm{i}\delta_2} & \cdots 0 \\ \vdots & \vdots & \ddots \vdots \\ 0 & 0 & \cdots \mathrm{e}^{2\mathrm{i}\delta_n} \end{pmatrix}. \tag{100}$$

Multiplying (98) by O:

$$\begin{cases} O\,(W_1+W_2)=S_{\mathrm{diag}}\,(O\,(W_1+W_2))^*, \\ O\,(W_1-W_2)=-S_{\mathrm{diag}}\,(O\,(W_1-W_2))^*, \end{cases} \tag{101}$$

or explicitly, in terms of components:

$$\begin{cases} [O\,(W_1+W_2)]_\alpha=\mathrm{e}^{2\mathrm{i}\delta_\alpha}\,[(O\,(W_1+W_2))]_\alpha^*, \\ [O\,(W_1-W_2)]_\alpha=-\mathrm{e}^{2\mathrm{i}\delta_\alpha}\,[(O\,(W_1-W_2))]_\alpha^*. \end{cases} \tag{102}$$

This implies that each $[O\,(W_1+W_2)]_\alpha$ is a complex number, with phase δ_α (and similarly for $[O\,(W_1-W_2)]_\alpha$):

$$\begin{cases} [O\,(W_1+W_2)]_\alpha\equiv 2\mathrm{e}^{\mathrm{i}\delta_\alpha}\,[R]_\alpha, \\ [O\,(W_1-W_2)]_\alpha\equiv 2\mathrm{i}\mathrm{e}^{\mathrm{i}\delta_\alpha}\,[I]_\alpha. \end{cases} \tag{103}$$

with R and I real (the factor 2 is unimportant). Solving for OW:

$$\begin{cases} [OW_1]_\alpha=\mathrm{e}^{\mathrm{i}\delta_\alpha}\,[R+\mathrm{i}I]_\alpha, \\ [OW_2]_\alpha=\mathrm{e}^{\mathrm{i}\delta_\alpha}\,[R-\mathrm{i}I]_\alpha. \end{cases} \tag{104}$$

This is the generalized Watson theorem.

Bare amplitudes The main point is to consider $[R\pm\mathrm{i}I]$ as weak amplitudes without final-state interactions, i.e. as bare amplitudes:

$$\begin{cases} [R+\mathrm{i}I]_\alpha=[OW_{1,\mathrm{b}}]_\alpha, \\ [R-\mathrm{i}I]_\alpha=[OW_{2,\mathrm{b}}]_\alpha. \end{cases} \tag{105}$$

So we see that $\mathrm{W}_{1,\mathrm{b}}$ and its CP conjugate $\mathrm{W}_{2,\mathrm{b}}$ are complex conjugate, as they should for weak amplitudes without FSI. In doing this, we put all the FSI effects in the phases, and therefore the norms are not modified. Expressing the full amplitudes in terms of these bare ones (see (105) and (104)), we get

$$\begin{cases} [OW_1]_\alpha=\mathrm{e}^{\mathrm{i}\delta_\alpha}\,[OW_{1,\mathrm{b}}]_\alpha \\ [OW_2]_\alpha=\mathrm{e}^{\mathrm{i}\delta_\alpha}\,[OW_{2,\mathrm{b}}]_\alpha \end{cases} \Leftrightarrow \begin{cases} OW_1=\sqrt{S_{\mathrm{diag}}}OW_{1,\mathrm{b}}, \\ OW_2=\sqrt{S_{\mathrm{diag}}}OW_{2,\mathrm{b}}; \end{cases} \tag{106}$$

$$W_i=\sqrt{S}W_{i,\mathrm{b}}, \tag{107}$$

where we identify the square root as

$$\sqrt{S}=O^{\mathrm{t}}\sqrt{S_{\mathrm{diag}}}O. \tag{108}$$

We have thus demonstrated the form (14), which is the generalized Watson theorem rewritten using bare amplitude identifications.

In summary, Watson's theorem allows us to extract from the full weak amplitudes the hadronic FSI part, leaving real bare amplitudes. We can say that we have unitarized weak bare decay amplitudes, since with the adjunction of the strong phases, the full **S**-matrix is unitary. We can also say that we have renormalized the weak bare amplitudes by $S^{1/2}$, i.e. the effect of FSI factorize.

5.1 Restriction on mixings

Neglecting some mixings, we can impose a block-diagonal form for S:

$$\begin{pmatrix} 1 & \mathrm{i}W_1^{\mathrm{t}} & \mathrm{i}Z_1^{\mathrm{t}} \\ \mathrm{i}W_2 & S_1 & 0 \\ \mathrm{i}Z_2 & 0 & S_2 \end{pmatrix}. \tag{109}$$

For each decoupled part, we can repeat the whole analysis. Indeed, unitarity implies

$$\begin{cases} S_1^\dagger S_1 = S_1 S_1^\dagger = 1, \\ W_1 = S_1 W_2^*, \\ W_2 = S_1 W_1^*, \end{cases} \quad \text{and} \quad \begin{cases} S_2^\dagger S_2 = S_2 S_2^\dagger = 1, \\ Z_1 = S_2 Z_2^*, \\ Z_2 = S_2 Z_1^*, \end{cases} \tag{110}$$

and the remaining discussion is straightforward.

The probability conservation allows a characterization of this approximation. It is now expressed, if S_1 is $n_1 \times n_1$ and S_2 is $n_2 \times n_2$ with $n = n_1 + n_2$:

$$\begin{aligned} &\sum_{i=1}^{n} \|(B \to x_i)\|^2 = \sum_{i=1}^{n} \|(B \to \{x_i\})\|^2 \\ &\to \begin{cases} \displaystyle\sum_{i=1}^{n_1} \|(B \to x_i)\|^2 = \sum_{i=1}^{n_1} \|(B \to \{x_i\})\|^2, \\ \displaystyle\sum_{i=n_1}^{n} \|(B \to x_i)\|^2 = \sum_{i=n_1}^{n} \|(B \to \{x_i\})\|^2. \end{cases} \end{aligned} \tag{111}$$

It remains to be seen in each case wether this is an appropriate restriction or not.

5.2 Mixing parameters

The most general mixings will be specified by a general orthogonal transformation O on M_{diag}. The determinant of this matrix can be ± 1, but we can restrict our attention to orthogonal matrices of determinant $+1$, and introduce a diagonal matrix P with diagonal element ± 1. This P matrix will always disappear when calculating M since $M = O^{\mathrm{t}} P M_{\mathrm{diag}} P O = O^{\mathrm{t}} M_{\mathrm{diag}} O$.

For example, a general two-channel mixing can be described from

$$O = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \quad \text{and} \quad M = O^{\mathrm{t}} M_{\mathrm{diag}} O, \tag{112}$$

where α is a mixing parameter. For a three-channel mixing, we will need tree mixing parameters (Euler's angles) and so on.

5.3 M-matrix properties

Because they were built as real orthogonal transformations on diagonal unitary matrices, the M-matrices have a number of properties.

(i) They are symmetric and unitary.
(ii) The nth power is trivial: just multiply all phases by n. This is also valid for n rational (see (108)).
(iii) When all the phases are equal to δ, M is simply $\mathrm{e}^{\mathrm{i}\delta}1$ since $O^{\mathrm{t}} 1 O = 1$. In other words, when all the eigenphases are equal, mixings disappear.
(iv) Finally, write M_{diag} as

$$\begin{aligned} M_{\mathrm{diag}} = \mathrm{e}^{\mathrm{i}\delta_1} 1 + \left(\mathrm{e}^{\mathrm{i}\delta_2} - \mathrm{e}^{\mathrm{i}\delta_1}\right) \begin{pmatrix} 0\,0\,0\cdots \\ 0\,1\,0\cdots \\ 0\,0\,0\cdots \\ \vdots\,\vdots\,\vdots\,\ddots \end{pmatrix} \\ + \left(\mathrm{e}^{\mathrm{i}\delta_3} - \mathrm{e}^{\mathrm{i}\delta_1}\right) \begin{pmatrix} 0\,0\,0\cdots \\ 0\,0\,0\cdots \\ 0\,0\,1\cdots \\ \vdots\,\vdots\,\vdots\,\ddots \end{pmatrix} + \cdots \end{aligned} \tag{113}$$

and apply the orthogonal transformation. In this way, we have decomposed M into a sum of numerical matrices, with phase differences as coefficients:

$$\begin{aligned} M &= \mathrm{e}^{\mathrm{i}\delta_1} 1 + \sum_{i=2}^{n} \left(\mathrm{e}^{\mathrm{i}\delta_i} - \mathrm{e}^{\mathrm{i}\delta_1}\right) A_i \\ &= \mathrm{e}^{\mathrm{i}\delta_1} \left(1 + \sum_{i=2}^{n} \left[\left(\mathrm{e}^{\mathrm{i}(\delta_i - \delta_1)} - 1\right) A_i\right]\right), \end{aligned} \tag{114}$$

with $A_i A_j = \delta_{ij} A_i$. We can, of course, factor another phase than $\mathrm{e}^{\mathrm{i}\delta_1}$. These forms can be useful phenomenologically.

6 Passage from a mixing parameter formulation towards an elasticity-parameter formulation.

The form for the **S**-matrix in terms of elasticity parameters are built in the following way.

Define the base transformations as

$$\left|\overrightarrow{C}\right\rangle = O(\alpha) \left|\overrightarrow{T}\right\rangle = O(\alpha) O_{\mathrm{SU}(2)} \left|\overrightarrow{X}\right\rangle = O(\beta) \left|\overrightarrow{X}\right\rangle, \tag{115}$$

with

$$O(\gamma = \alpha, \beta) = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}.$$

From this, we can directly write the link between the $S_{\mathrm{eigen}}, S_{\mathrm{isospin}}$ and S_{physical}:

$$\begin{cases} S_{\mathrm{isospin}}(\delta_1, \delta_2, \alpha) = O^{\mathrm{t}}(\alpha) S_{\mathrm{eigen}} O(\alpha), \\ S_{\mathrm{physical}}(\delta_1, \delta_2, \beta) = O^{\mathrm{t}}(\beta) S_{\mathrm{eigen}} O(\beta), \end{cases} \tag{116}$$

and $S_{\mathrm{physical}} = O^{\mathrm{t}}_{\mathrm{SU}(2)} S_{\mathrm{isospin}} O_{\mathrm{SU}(2)}$, where we have explicitly written the parameters: the eigenphases and the mixing parameters.

In these last expressions, we will change the parameters from:

$$\begin{cases} \delta_1, \delta_2, \alpha \to w_1, w_2, \eta_{\mathrm{w}}, \\ \delta_1, \delta_2, \beta \to \alpha_1, \alpha_2, \eta. \end{cases} \tag{117}$$

The elasticity parameters are defined in terms of eigenphases and mixing parameters as

$$\begin{cases} \eta_{\rm w} = \sqrt{1 - 4\varepsilon_w^2 \sin^2(\delta_2 - \delta_1)}; & \varepsilon_w = \cos\alpha \sin\alpha, \\ \eta = \sqrt{1 - 4\varepsilon^2 \sin^2(\delta_2 - \delta_1)}; & \varepsilon = \cos\beta \sin\beta. \end{cases} \tag{118}$$

For the phases, we have expressions like

$$\begin{cases} 2\alpha_1 = \arg\left(\cos^2\beta e^{2i\delta_1} + \sin^2\beta e^{2i\delta_2}\right), \\ 2\alpha_2 = \arg\left(\sin^2\beta e^{2i\delta_1} + \cos^2\beta e^{2i\delta_2}\right), \end{cases} \tag{119}$$

and similarly for w_1, w_2 in $S_{\rm isospin}$.

Finally, the different limits for the passage $\delta_1, \delta_2, \beta \longleftrightarrow \alpha_1, \alpha_2, \eta$ are

$$\begin{cases} \beta = 0 \Rightarrow \eta = 1, \alpha_1 = \delta_1, \alpha_2 = \delta_2; \\ \delta_2 = \delta_1 \Rightarrow \alpha_1 = \alpha_2, \eta = 0. \end{cases} \tag{120}$$

We can also characterize the *maximal mixing*: The limit $\alpha_1 = \alpha_2, \eta \neq 0$ can be obtained with $\beta = 45°$; this gives the smallest value for η for a given $\delta_2 - \delta_1$, i.e. $\eta = \cos(\delta_2 - \delta_1)$.

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